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On the existence of kernels and h-kernels in directed graphs

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Abstract

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A directed graph D with vertex set V is called cyclically h-partite ($h \geq 2$) provided one can partition $V = V_0 + V_1 + \cdots + V_{h-1}$ so that if (u, v) is an arc of D then $u \in V_i$, and $v \in V_{i+1}$ (notation mod h). In this communication we obtain a characterization of cyclically h-partite strongly connected digraphs. As a consequence we obtain a sufficient condition for a digraph to have a h-kernel.

Introduction

For general concepts we refer the reader to [1]. Let D be a digraph; $V(D)$ and $A(D)$ will denote the sets of vertices and arcs of D respectively. An arc $u_1u_2 \in A(D)$ is called asymmetrical (resp. symmetrical) if $u_2u_1 \notin A(D)$ (resp. $u_2u_1 \in A(D)$). If $S \subseteq V(D)$ $D[S]$ will denote the subdigraph of D induced by S. The asymmetrical part of D (resp. symmetrical part of D), which is denoted by Asym D (resp. *Sym* D) is the spanning subdigraph of D whose arcs are the asymmetrical (resp. symmetrical) arcs of D.

If $\mathscr{C}$ is a directed walk we will denote by $l(\mathscr{C})$ its length. By the directed distance $d_D(x, y)$ from the vertex x to vertex y in a digraph D we mean the length of the shortest directed path from x to y in D.

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A circuit of D is a sequence of vertices of D, $\mathscr{C} = (0, 1, \ldots, n-1)$ such that $(i, i+1) \in A(D)$ (notation $\bmod n$).

Let h be a natural number with $h \geq 2$. A set $J \subseteq V(D)$ will be called a h-kernel of the digraph D if:

(1) For reach $x, x' \in J$, $x \neq x'$ we have $d_D(x, x') \geq h$.

(2) For each $y \in (V(D) - J)$, there exists $x \in J$ such that $d_D(y, x) \leq h-1$.

For $h = 2$ we have a kernel in the sense of Berge [1]. When every induced subdigraph of D has a kernel, D is said to be kernel perfect or KP-digraph. Note that the h-kernel of D is the kernel of a digraph D^h obtained from D by putting an arc xy if there is in D a directed path from x to y of length $\leq h-1$.

1. A characterization of cyclically h-partite strongly connected digraphs

Definition 1.1. A digraph D is called cyclically h-partite ($h \geq 2$) provided one can partition $V(D) = V_0 + V_1 + \cdots + V_{h-1}$ so that if (u, v) is an arc of D then $u \in V_i$, and $v \in V_{i+1}$ (notation $\bmod h$).

In case $h = 2$ we obtain the bipartite digraphs in the usual sense.

Lemma 1.1 [3]. *Every closed directed walk of length* $\not\equiv 0 \pmod h$, $h \geq 2$, *contains a circuit of length* $\not\equiv 0 \pmod h$.

Theorem 1.1. *Let D be a strongly connected digraph, D is a cyclically h-partite digraph if and only if there exists a strongly connected subdigraph $H \subseteq D$ such that every circuit of length $\not\equiv 0 \pmod h$ has at least two arcs in $A(D) - A(H)$.*

Proof. It is clear from Definition 1.1 that if D is a cyclically h-partite digraph then every circuit of D has length $\equiv 0 \pmod h$. So when D is a strongly connected cyclically h-partite digraph, the subdigraph $H = D$ satisfies the required properties.

Suppose that there exists a strongly connected subdigraph $H \subseteq D$ such that every circuit of length $\not\equiv 0 \pmod h$ has at least two arcs in $A(D) - A(H)$.

Let $m_0 \in V(D)$ and for each $0 \leq i < h$ let $N_i \subseteq V(D)$ defined as follows:

$$N_i = \{z \in V(D) \mid \text{there exists an } m_0z\text{-directed walk contained in } H \text{ of length} \equiv i \pmod h\}.$$

(i) We claim that $N_i \cap N_j = \emptyset$ for $i \neq j$, $i, j \in \{0, 1, \ldots, h-1\}$. If $z \in N_i \cap N_j$ then there exists an m_0z-directed walk T_1 contained in H, such that $l(T_1) \equiv i \pmod h$ and there exists an m_0z-directed walk T_2 contained in H with $l(T_2) \equiv j \pmod h$. Since H is strongly connected there exists a zm_0-directed walk T_3 contained in H; $T_1 \cup T_3$ and $T_2 \cup T_3$ are closed directed walks contained in H so it follows from Lemma 1.1 and the hypothesis that $l(T_1 \cup T_3) \equiv l(T_2 \cup T_3) \equiv 0$ (mod h which is impossible.

(ii) We claim that each N_i is an independent set of D (i.e. there is no arc of D with both terminal endpoints in N_i).

Let $x, y \in N_i$ and suppose that $(x, y) \in A(D)$. Let T_x be an m_0x-directed walk contained in H, T_y an m_0y-directed walk contained in H such that $l(T_x) \equiv l(T_y) \equiv i \pmod h$ and T a ym_0-directed walk contained in H. Since $T_y \cup T$ is a closed directed walk contained in H and $T_x \cup (x, y) \cup T$ is a closed directed walk with at most one arc in $A(D) - A(H)$ it follows from Lemma 1.1 and the hypothesis that $l(T_x \cup T) \equiv l(T_y \cup T) \equiv 0 \pmod h$ and $l(T_x \cup (x, y) \cup T) \equiv 0 \pmod h$ and hence $l(T_x \cup (x, y) \cup T) \equiv l(T_x \cup T) + 1 \equiv 1 \pmod h$ which is impossible.

(iii) Every arc with initial endpoint in N_i has terminal endpoint in N_{i+1} (notation $\bmod h$) and $V(D) = \bigcup_{i=0}^{h-1} N_i$. Let (x, y) an arc with initial endpoint in N_i, it follows from (ii) that $y \in N_j$ for some $j \in \{0, 1, \ldots, h-1\} - \{i\}$. Let T_x an m_0x-directed walk contained in H with $l(T_x) \equiv i \pmod h$, T_y an m_0y-directed walk contained in H with $l(T_y) \equiv j \pmod h$ and T a ym_0-directed walk contained in H. We have $T_x \cup (x, y) \cup T$ is a closed directed walk with at most one arc in $V(D) - V(H)$ so it follows from Lemma 1.1 and the hypothesis that $l(T_x \cup (x, y) \cup T) \equiv 0 \pmod h$ also we have $l(T_y \cup T) \equiv 0 \pmod h$ hence $l(T_x) + 1 \equiv l(T_y) \pmod h$ and $j \equiv i + 1 \pmod h$ and because $j \in \{0, \ldots, h-1\}$ it follows that $j = i + 1$. We conclude from (i), (ii), (iii) and Definition 1.1 that D is cyclically h-partite. □

As a direct consequence of Theorem 1.1 we have the following result.

Theorem 1.2. *Let D be a digraph, $h \geq 2$. If H is a subdigraph of D such that every circuit of length* $\not\equiv 0 \pmod h$ *has at least two arcs in $V(D) - V(H)$ then for each strong component α of H the digraph $D[V(\alpha)]$ is a cyclically h-partite digraph.*

Corollary 1.1. *Let D be a digraph. If H is a subdigraph of D such that every odd circuit has at least two arcs in $V(D) - V(H)$ then for each strong component $\mathscr{C}$ of H the digraph $D[V(\mathscr{C})]$ is a bipartite digraph.*

Corollary 1.2. *Let D be a strongly connected digraph.*

D is a bipartite digraph if and only if there exists an strongly connected subdigraph $H \subseteq D$ such that every odd circuit has at least two arcs in $A(D) - A(H)$.

2. On the existence of h-kernels in digraphs

Most of this section is based on Theorem 2.1.

Theorem 2.1. *Let D be a digraph. If there exists a strongly connected subdigraph $H \subseteq D$ such that every circuit of length* $\not\equiv 0 \pmod h$ *has at least two arcs in $A(D) - A(H)$ then D has a h-kernel.*

Proof. It follows from Theorem 1.1 that D is cyclically h-partite strongly connected digraph, and if $V(D) = N_0 + \cdots + N_{h-1}$ is a h-partition of $V(D)$ it follows directly from Definition 1.1 that each N_i is a h-kernel of D.

Theorem 2.2. *Let D be a digraph such that* Asym(D) *is strongly connected. If every circuit of length* $\not\equiv 0 \pmod h$ *has at least two symmetrical arcs then D has a h-kernel.*

As a direct consequence we obtain the following result.

Theorem 2.3 [3]. *Let D be a digraph such that* Asym(D) *is strongly connected. Furthermore suppose that for every circuit γ such that* $l(\gamma) \not\equiv 0 \pmod h$ *either* (a) *or* (b) *is satisfied.*

(a) *Every arc of γ is a symmetrical arc of D.*

(b) *γ has at least h symmetrical arcs.*

Then D has a h-kernel.

Corollary 2.1 [5]. *Let D be a strongly connected digraph such that every circuit of D has length* $\equiv 0 \pmod h$, $h \geq 2$. *Then D has a h-kernel.*

Corollary 2.2. *Let D be a digraph. If there exists a strongly connected subdigraph $H \subseteq D$ such that every odd circuit has at least two arcs in $A(D) - A(H)$ then D is kernel-perfect.*

Proof. It follows from Corollary 1.2 that D is a bipartite strongly connected digraph and it is a well-known result that bipartite digraphs are kernel-perfect.

Another consequence of Theorem 2.1 is the following result due to Duchet.

Theorem 2.4. *Let D be a digraph. If every odd circuit in D has at least two symmetrical arcs, then D is kernel-perfect.*

Proof. It follows from Corollary 1.1 that for each strong component α of Asym(D) the digraph $D[V(\alpha)]$ is a bipartite digraph and Theorem 2.4 follows from the following observation.

Observation 2.1 [4]. If for every strong component α of Asym(D), $D[V(\alpha)]$ is bipartite then D is a kernel-perfect digraph.

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References

[1] C. Berge, Graphs and Hypergraphs (North-Holland, Amsterdam, 1976).
[2] P. Duchet, Graphes noyau-parfaits, Ann. Discrete Math. 9 (1980) 93–101.
[3] H. Galeana-Sánchez, On the existence of (K, l)-kernels in digraphs, Discrete Math. 85 (1990) 99–102.
[4] H. Galeana-Sánchez and V. Neumann-Lara, On kernel perfect critical digraphs Discrete Math. 59 (1986) 257–265.
[5] M. Kwaśnik, The generalization of Richardson theorem, Discussiones Math. IV (1981) 11–14.