On common generalizations of both tournaments and bipartite tournaments

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Abstract

In [1] and [3], Jørgen Bang-Jensen introduced five new classes of digraphs as a common generalizations of both tournaments and bipartite tournaments and conjectured that these classes of digraphs generalize a known result about bipartite tournaments: a bipartite tournament is Hamiltonian if and only if it is strong and has a cycle factor.

Our work, together with work of Wang and Wang, has led to set the conjecture true.

Tournaments and multipartite tournaments

Tournaments (orientations of complete digraphs) are without doubt the best studied class of digraphs (Bang-Jensen and Gutin, and Volkmann).

Multipartite tournaments (orientations fo complete multipartite graphs) are newly defined classes of graphs on which we have a lot of results.

Semicomplete digraphs are a slighty generalization of tournaments. Semicomplete digraphs admits simmetric arcs, *i.e.*, between two vertices can be two arcs in opposite directions.

Analougously, ***semicomplete multipartite digraphs*** are a slighty generalization of multipartite tournaments. Semicomplete multipartite digraphs admits simmetric arcs.

Generalizations of tournaments

In [2], Bang-Jensen proposed to work in classes of digraphs which have tournaments as a proper class for the purpose of extending the well known results about tournaments. He called such classes of digraphs ***generalizations of tournaments***.

Local tournaments

Local tournaments was the first class of digraphs defined in the framework of generalizations of tournaments. This class was defined by Bang-Jensen in [2]. A digraphs is said to be a ***local tournament*** if both the in-neighborhood and the out-neighborhood of each vertex induce a tournament.

Tournaments are examples of local tournaments as well as any cycle.

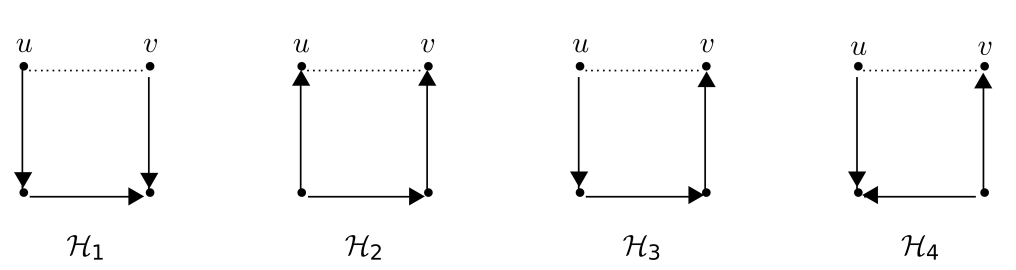
The class of local tournaments has been a great generalization of tournaments since it isn't a trivial generalization and a lot of results about tournaments naturally hold in this class.

Common generalizations of both tournaments and bipartite tournaments

In [1], Bang-Jensen looked for an analogous class to local tournaments for bipartite tournaments. He defined ***arc-locally semicomplete*** digraphs as a common generalization of both tournaments and bipartite tournaments. Later, he improved his idea to define $\mathcal{H}_i$-free digraphs, for $i = 1, 2, 3$ and 4.

Arc-locally semicomplete digraphs and $\mathcal{H}_i$-free digraphs

Let us think in the following digraphs:



$\mathcal{H}_i$-free digraphs

Definition 1. *We say that a digraph is **$\mathcal{H}_i$-free**, if u and v are always adjacent vertices wherever they appear in a configuration as $\mathcal{H}_i$.*

Definition 2. *We say that a digraph is **arc-locally semicomplete**, if D is a $\{\mathcal{H}_1, \mathcal{H}_2\}$-free digraph.*

Hamiltonicity in tournaments and bipartite tournaments

As we all know, every tournament is Hamiltonian if and only if it is strong.

What happens to bipartite tournaments? Unfortunately, strongness is a necessary condition but it is not a sufficient condition. There is a nice characterization of Hamiltonicity in bipartite tournaments.

Definition 3. *A **cycle factor** in a digraph D is a collection of vertex disjoint cycles which cover $V(D)$.*

Theorem 1 (Gutin, and Häggkvist and Manoussakis)**.** *A bipartite tournament is Hamiltonian if and only if it is strong and has a cycle factor.*

Since $\mathcal{H}_i$-free digraphs are generalizations of both tournaments and bipartite tournaments, Bang-Jensen conjectured that:

Conjecture 1. *A $\mathcal{H}_i$-free digraph, for $i = 1, 2, 3$ and 4, is Hamiltonian if and only if it is strong and has a cycle factor.*

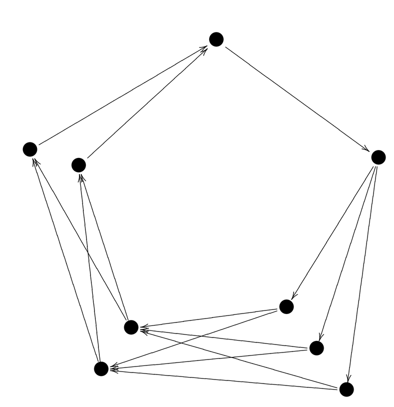
From this conjecture, research about $\mathcal{H}_i$-free digraphs has been focused on characterizing these classes of digraphs or, at least, on giving structural properties about these classes.

Results about $\mathcal{H}_i$-free digraphs

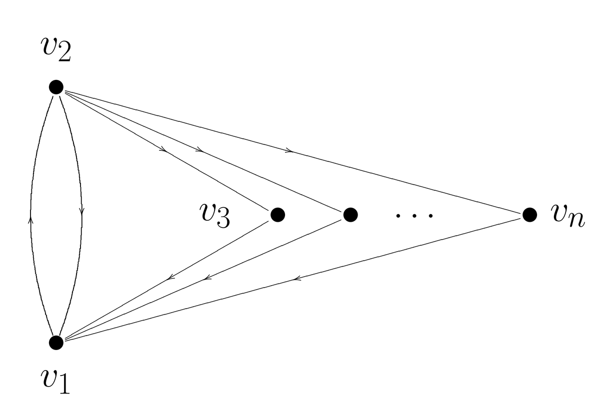
Arc-locally semicomplete digraphs ($\{\mathcal{H}_1, \mathcal{H}_2\}$-free digraphs)

First, let us define two new classes of digraphs.

Definition 4. *An **extended cycle** is a digraph which can be obtained from a cycle C by substituting an independent set for each vertex of C.*



Definition 5. *For each $n \geq 4$, let W_n be the digraph with vertex sex $V(D) = \{v_1, v_2, \ldots, v_n\}$ and arc set $A(D) = \{v_1v_2, v_2v_1, v_2v_3, v_3v_1\} \cup \{v_2v_i, v_iv_1 : 4 \leq i \leq n\}$.*



Theorem 2 (Galeana-Sánchez and Goldfeder [4, 6], Wang and Wang [8])**.** *If D is a strong arc-locally semicomplete digraph, then D is either a semicomplete digraph, a semicomplete bipartite digraph, an extended cycle, or W_n, for some $n \geq 4$.*

In [4], we provided a characterization of arc-locally semicomplete digraphs (not only strong) in terms of simpler families of digraphs.

As a consequence, we have:

Corollary 1. *A $\{\mathcal{H}_1, \mathcal{H}_2\}$-free digraph is Hamiltonian if and only if it is strong and has a cycle factor.*

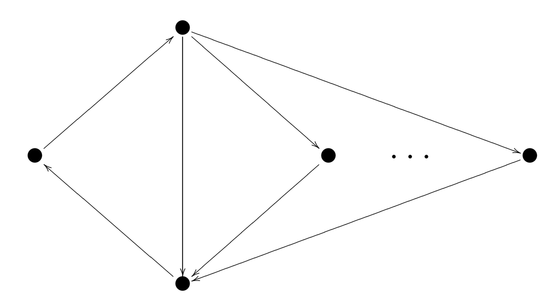
In [8], Wang and Wang provide a characterization of strong $\{\mathcal{H}_1\}$-free digraphs. As a consequence, they obtained:

Corollary 2. *A $\mathcal{H}_i$-free digraph, for $i = 1, 2$, is Hamiltonian if and only if it is strong and has a cycle factor.*

$\mathcal{H}_3$-free digraphs

Let us define a new classes of digraphs.

Definition 6. *For each $n \geq 4$, let be F_n the digraph with vertex set $V(D) = \{x_1, x_2, \ldots, x_n\}$ and arc set $A(D) = \{x_1x_2, x_2x_3, x_3x_1\} \cup \{x_1x_i, x_ix_2 : 4 \leq i \leq n\}$.*



In [7], Galeana-Sánchez, Goldfeder and Urrutia gave a characterization of strong $\mathcal{H}_3$-free digraphs:

Theorem 3 (Galeana-Sánchez, Goldfeder and Urrutia [7])**.** *Let D be a strong 3-quasi-transitive digraph of order n. Then D is either a semicomplete digraph, a semicomplete bipartite digraph, or isomorphic to F_n.*

As a consequence, we have:

Corollary 3. *A $\mathcal{H}_3$-free digraph is Hamiltonian if and only if it is strong and has a cycle factor.*

$\mathcal{H}_4$-free digraphs

Giving a characterization of strong $\mathcal{H}_4$-free digraphs seems a very difficult task. However, we were be able to prove Bang-Jensen's conjecture for this class of digraphs.

Theorem 4 (Galena-Sánchez and Goldfeder [5])**.** *A $\mathcal{H}_4$-free digraph is Hamiltonian if and only if it is strong and has a cycle factor.*

Conclusions

Bang-Jensen's conjecture is true: $\mathcal{H}_i$-free digraphs generalizes the corresponding theorem for bipartite tournaments. However, these classes of digraphs are still very near to bipartite tournaments, it is unlike the situation between local tournaments and tournaments.

The question is still open: **Is there a generalization of semicomplete tournaments as local tournaments generalizes tournaments?**

References

[1] J. Bang-Jensen. Arc-local tournament digraphs: a generalization of tournaments and bipartite tournaments. Technical Report Preprint no. 2, Department of Mathematics and Computer Science, University of Southern Denmark, 1993.

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