Chapter 11

Monochromatic and Rainbow Colorings

There are instances in which we will be interested in edge colorings of graphs that do not require adjacent edges to be assigned distinct colors. Of course, in these cases such colorings are not proper edge colorings. In many of these instances, we are concerned with coloring the edges of complete graphs. There are occasions when we have a fixed number of colors and seek the smallest order of a complete graph such that each edge coloring of this graph with this number of colors results in a prescribed subgraph all of whose edges are colored the same (a monochromatic subgraph). However, for a fixed graph (complete or not), we are also interested in the largest number of colors in an edge coloring of the graph that avoids both a specified subgraph all of whose edges are colored the same and another specified subgraph all of whose edges are colored differently (a rainbow subgraph). The best known problems of these types deal with the topic of Ramsey numbers of graphs. We begin with this.

11.1 Ramsey Numbers

Frank Plumpton Ramsey (1903–1930) was a British philosopher, economist, and mathematician. Ramsey’s first major work was his 1925 paper “The Foundations of Mathematics”, in which he intended to improve upon Principia Mathematica by Bertrand Russell and Alfred North Whitehead. He presented his second paper “On a problem of formal logic” [144] to the London Mathematical Society. A restricted version of this theorem is the following.

**Theorem 11.1 (Ramsey’s Theorem)**  For any \( k + 1 \geq 3 \) positive integers \( t, n_1, n_2, \ldots, n_k \), there exists a positive integer \( N \) such that if each of the \( t \)-element subsets of the set \( \{1, 2, \ldots, N\} \) is colored with one of the \( k \) colors 1, 2, \ldots, \( k \), then for some integer \( i \) with \( 1 \leq i \leq k \), there is a subset \( S \) of \( \{1, 2, \ldots, N\} \) containing \( n_i \) elements such that every \( t \)-element subset of \( S \) is colored \( i \).
In order to see the connection of Ramsey’s theorem with graph theory, suppose that \( \{1, 2, \ldots, N\} \) is the vertex set of the complete graph \( K_N \). In the case where \( t = 2 \), each 2-element subset of the set \( \{1, 2, \ldots, N\} \) is assigned one of the colors \( 1, 2, \ldots, k \), that is, there is a \( k \)-edge coloring of \( K_N \). It is this case of Ramsey’s theorem in which we have a special interest.

**Theorem 11.2 (Ramsey’s Theorem)** For any \( k \geq 2 \) positive integers \( n_1, n_2, \ldots, n_k \), there exists a positive integer \( N \) such that for every \( k \)-edge coloring of \( K_N \), there is a complete subgraph \( K_{n_i} \) of \( K_N \) for some \( i \) (1 \( \leq i \leq k \)) such that every edge of \( K_{n_i} \) is colored \( i \).

In fact, our primary interest in Ramsey’s theorem is the case where \( k = 2 \). In a red-blue edge coloring (or simply a red-blue coloring) of a graph \( G \), every edge of \( G \) is colored red or blue. Adjacent edges may be colored the same; in fact, this is often necessary. Indeed, in a red-blue coloring of \( G \), it is possible that all edges are colored red or all edges are colored blue. For two graphs \( F \) and \( H \), the Ramsey number \( R(F, H) \) is the minimum order \( n \) of a complete graph such that for every red-blue coloring of \( K_n \), there is either a subgraph isomorphic to \( F \) all of whose edges are colored red (a red \( F \)) or a subgraph isomorphic to \( H \) all of whose edges are colored blue (a blue \( H \)). Certainly

\[
R(F, H) = R(H, F)
\]

for every two graphs \( F \) and \( H \). The Ramsey number \( R(F, F) \) is thus the minimum order \( n \) of a complete graph such that if every edge of \( K_n \) is colored with one of two given colors, then there is a subgraph isomorphic to \( F \) all of whose edges are colored the same (a monochromatic \( F \)). The Ramsey number \( R(F, F) \) is sometimes called the Ramsey number of the graph \( F \). We begin with perhaps the best known Ramsey number, namely the Ramsey number of \( K_3 \).

**Theorem 11.3** \( R(K_3, K_3) = 6 \).

**Proof.** Let there be given a red-blue coloring of \( K_6 \). Consider some vertex \( v_1 \) of \( K_6 \). Since \( v_1 \) is incident with five edges, it follows by the Pigeonhole Principle that at least three of these five edges are colored the same, say red. Suppose that \( v_1v_2, v_1v_3, \) and \( v_1v_4 \) are red edges. If any of the edges \( v_2v_3, v_2v_4, \) and \( v_3v_4 \) is colored red, then we have a red \( K_3 \); otherwise, all three of these edges are colored blue, producing a blue \( K_3 \). Hence \( R(K_3, K_3) \leq 6 \). On the other hand, let \( V(K_5) = \{v_1, v_2, \ldots, v_5\} \) and define a red-blue coloring of \( K_5 \) by coloring each edge of the 5-cycle \( (v_1, v_2, \ldots, v_5, v_1) \) red and the remaining edges blue (see Figure 11.1, where bold edges are colored red and dashed edges are colored blue). Since this red-blue coloring produces neither a red \( K_3 \) nor a blue \( K_3 \), it follows that \( R(K_3, K_3) \geq 6 \) and so \( R(K_3, K_3) = 6 \).

Theorem 11.3 provides the answer to a popular question. In any gathering of people, every two people are either acquaintances or strangers. What is the smallest positive integer \( n \) such that in any gathering of \( n \) people, there are either three mutual acquaintances or three mutual strangers? This situation can be modeled
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by a graph of order \( n \), in fact by \( K_n \), where the vertices are the people, together
with a red-blue coloring of \( K_n \), where a red edge, say, indicates that the two people
are acquaintances and a blue edge indicates that the two people are strangers. By
Theorem 11.3 the answer to the question asked above is 6.

As an example of a Ramsey number \( R(F, H) \), where neither \( F \) nor \( H \) is complete,
we determine \( R(F, H) \) for the graphs \( F \) and \( H \) shown in Figure 11.2.

\[ \begin{align*}
F : & \quad H :
\end{align*} \]

Figure 11.2: Determining \( R(F, H) \)

**Example 11.4** For the graphs \( F \) and \( H \) shown in Figure 11.2,

\[ R(F, H) = 7. \]

**Proof.** Since the red-blue coloring of \( K_6 \) in which the red graph is \( 2K_3 \) and the
blue graph is \( K_{3,3} \) has neither a red \( F \) nor a blue \( H \), it follows that \( R(F, H) \geq 7 \). Now
let there be given a red-blue coloring of \( K_7 \). Since \( R(K_3, K_3) = 6 \) by Theorem 11.3,
\( K_7 \) contains a monochromatic \( K_3 \). Let \( U \) be the vertex set of a monochromatic \( K_3 \)
and let \( W \) be the set consisting of the remaining four vertices of \( K_7 \). We consider
two cases.

**Case 1.** The monochromatic \( K_3 \) with vertex set \( U \) is blue. If any edge joining
two vertices of \( W \) is blue, then there is a blue \( H \); otherwise, there is a red \( F \).

**Case 2.** The monochromatic \( K_3 \) with vertex set \( U \) is red. If any edge joining a
vertex of \( U \) and a vertex of \( W \) is red, then there is a red \( F \). Otherwise, every edge
joining a vertex of \( U \) and a vertex of \( W \) is blue. If any edge joining two vertices of
\( W \) is blue, then there is a blue \( H \); otherwise, there is a red \( F \).

The Ramsey number \( R(F, H) \) of two graphs \( F \) and \( H \) can be defined without
regard to edge colorings. The Ramsey number \( R(F, H) \) can be defined as the
smallest positive integer \( n \) such that for every graph \( G \) of order \( n \), either \( G \) contains a subgraph isomorphic to \( F \) or its complement \( \overline{G} \) contains a subgraph isomorphic to \( H \). Assigning the color red to each edge of \( G \) and the color blue to each edge of \( \overline{G} \) returns us to our initial definition of \( R(F, H) \).

Historically, it is the Ramsey numbers \( R(K_s, K_t) \) that were the first to be studied. The numbers \( R(K_s, K_t) \) are commonly expressed as \( R(s, t) \) as well and are referred to as the classical Ramsey numbers. By Ramsey’s theorem, \( R(s, t) \) exists for every two positive integers \( s \) and \( t \). We begin with some observations.

First, as observed above, \( R(s, t) = R(t, s) \) for every two positive integers \( s \) and \( t \). Also, \( R(1, t) = 1 \) and \( R(2, t) = t \) for every positive integer \( t \); and by Theorem 11.3, \( R(3, 3) = 6 \).

Indeed, the Ramsey number \( R(F, H) \) exists for every two graphs \( F \) and \( H \). In fact, if \( F \) has order \( s \) and \( H \) has order \( t \), then \( R(F, H) \leq R(s, t) \).

The existence of the Ramsey numbers \( R(s, t) \) was also established in 1935 by Paul Erdős and George Szekeres [64], where an upper bound for \( R(s, t) \) was obtained as well. Recall, for positive integers \( k \) and \( n \) with \( k \leq n \), the combinatorial identity:

\[
\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.
\]  

(11.1)

**Theorem 11.5** For every two positive integers \( s \) and \( t \), the Ramsey number \( R(s, t) \) exists; in fact,

\[
R(s, t) \leq \binom{s + t - 2}{s - 1}.
\]

**Proof.** We proceed by induction on \( n = s + t \). We have already observed that \( R(1, t) = 1 \) and \( R(2, t) = t \) for every positive integer \( t \). Hence \( R(s, t) \leq \binom{s + t - 2}{s - 1} \) when \( n = s + t \leq 5 \). Thus we may assume that \( s \geq 3 \) and \( t \geq 3 \) and so \( n \geq 6 \). Suppose that \( R(s', t') \) exists for all positive integers \( s' \) and \( t' \) such that \( s' + t' < k \) where \( k \geq 6 \) and that

\[
R(s', t') \leq \binom{s' + t' - 2}{s' - 1}.
\]

We show for positive integers \( s \) and \( t \) with \( s, t \geq 3 \) and \( k = s + t \) that

\[
R(s, t) \leq \binom{s + t - 2}{s - 1}.
\]

By the induction hypothesis, the Ramsey numbers \( R(s - 1, t) \) and \( R(s, t - 1) \) exist and

\[
R(s - 1, t) \leq \binom{s + t - 3}{s - 2} \quad \text{and} \quad R(s, t - 1) \leq \binom{s + t - 3}{s - 1}.
\]
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Since 
\[
\binom{s + t - 3}{s - 2} + \binom{s + t - 3}{s - 1} = \binom{s + t - 2}{s - 1}
\]
by (11.1), it follows that 
\[
R(s - 1, t) + R(s, t - 1) \leq \binom{s + t - 2}{s - 1}.
\]

Let there be given a red-blue coloring of \(K_n\), where \(n = R(s - 1, t) + R(s, t - 1)\). We show that \(K_n\) contains either a red \(K_s\) or a blue \(K_t\). Let \(v\) be a vertex of \(K_n\). Then the degree of \(v\) in \(K_n\) is 
\[
p = R(s - 1, t) + R(s, t - 1) - 1.
\]
Let \(G\) be the spanning subgraph of \(K_n\) all of whose edges are colored red. Then \(\overline{G}\) is the spanning subgraph of \(K_n\) all of whose edges are colored blue. We consider two cases, depending on the degree of \(v\) in \(G\).

Case 1. \(\deg_G v \geq R(s - 1, t)\). Let \(A\) be the set of vertices in \(G\) that are adjacent to \(v\). Thus the order of the (complete) subgraph of \(K_n\) induced by \(A\) is 
\[
p = \deg_G v \geq R(s - 1, t).
\]
Hence this complete subgraph \(K_p\) contains either a red \(K_{s-1}\) or a blue \(K_t\). If \(K_p\) contains a blue \(K_t\), then \(K_n\) contains a blue \(K_t\) as well. On the other hand, if \(K_p\) contains a red \(K_{s-1}\), then \(K_n\) contains a red \(K_s\) since \(v\) is joined to every vertex of \(A\) by a red edge.

Case 2. \(\deg_G v \leq R(s - 1, t) - 1\). Then \(\deg_{\overline{G}} v \geq R(s, t - 1)\). Let \(B\) be the set of vertices in \(\overline{G}\) that are adjacent to \(v\). Therefore, the order of the (complete) subgraph of \(K_n\) induced by \(B\) is 
\[
q = \deg_{\overline{G}} v \geq R(s, t - 1).
\]
Hence this complete subgraph \(K_q\) contains either a red \(K_s\) or a blue \(K_{t-1}\). If \(K_q\) contains a red \(K_{s}\), then so does \(K_n\). If \(K_q\) contains a blue \(K_{t-1}\), then \(K_n\) contains a blue \(K_t\) since \(v\) is joined to every vertex of \(B\) by a blue edge.

Therefore, 
\[
R(s, t) \leq R(s - 1, t) + R(s, t - 1) \leq \binom{s + t - 2}{s - 1},
\]
completing the proof.

The proof of the preceding theorem provides an upper bound for \(R(s, t)\), which is, in general, an improvement to that stated in Theorem 11.5.

**Corollary 11.6** For integers \(s, t \geq 2\),
\[
R(s, t) \leq R(s - 1, t) + R(s, t - 1).
\]
(11.2)
Furthermore, if \(R(s - 1, t)\) and \(R(s, t - 1)\) are both even, then 
\[
R(s, t) \leq R(s - 1, t) + R(s, t - 1) - 1.
\]

**Proof.** The inequality in (11.2) follows from the proof of Theorem 11.5. Suppose that \(R(s - 1, t)\) and \(R(s, t - 1)\) are both even, and for
\[
n = R(s - 1, t) + R(s, t - 1),
\]

let there be given a red-blue coloring of \( K_{n-1} \). Let \( G \) be the spanning subgraph of \( K_{n-1} \) all of whose edges are colored red. Then every edge of \( G \) is blue. Since \( G \) has odd order, some vertex \( v \) of \( G \) has even degree. If \( \deg_G v \geq R(s - 1, t) \), then, proceeding as in the proof of Theorem 11.5, \( K_{n-1} \) contains a red \( K_s \) or a blue \( K_t \). Otherwise, \( \deg_G v \leq R(s - 1, t) - 2 \) and so \( \deg_G v \geq R(s, t - 1) \). Again, proceeding as in the proof of Theorem 11.5, \( K_{n-1} \) contains a red \( K_s \) or a blue \( K_t \). \( \blacksquare \)

Relatively few classical Ramsey numbers \( R(s, t) \) are known for \( s, t \geq 3 \). The table in Figure 11.3, constructed by Stanislaw Radziszowski [143], gives the known values of \( R(s, t) \) for integers \( s \) and \( t \) with \( s, t \geq 3 \) as of August 1, 2006.

<table>
<thead>
<tr>
<th>s</th>
<th>t</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
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<td>9</td>
<td>14</td>
<td>18</td>
<td>23</td>
<td>28</td>
<td>36</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>9</td>
<td>18</td>
<td>25</td>
<td>?</td>
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<td>?</td>
<td>?</td>
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</tbody>
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Figure 11.3: Some classical Ramsey numbers

While determining \( R(F, H) \) is challenging in most instances, Vašek Chvátal [45] found the exact value of \( R(F, H) \) whenever \( F \) is any tree and \( H \) is any complete graph.

**Theorem 11.7** Let \( T \) be a tree of order \( p \geq 2 \). For every integer \( n \geq 2 \),

\[
R(T, K_n) = (p - 1)(n - 1) + 1.
\]

**Proof.** First, we show that \( R(T, K_n) \geq (p - 1)(n - 1) + 1 \). Let there be given a red-blue coloring of the complete graph \( K_{(p-1)(n-1)} \) such that the resulting red subgraph is \( (n-1)K_{p-1} \); that is, the red subgraph consists of \( n-1 \) copies of \( K_{p-1} \). Since each component of the red subgraph has order \( p-1 \), it contains no connected subgraph of order greater than \( p-1 \). In particular, there is no red tree of order \( p \).

The blue subgraph is then the complete \((n-1)\)-partite graph \( K_{p-1,p-1,\ldots,p-1} \), where every partite set contains exactly \( p-1 \) vertices. Hence there is no blue \( K_n \) either. Since this red-blue coloring avoids both a red tree \( T \) and a blue \( K_n \), it follows that \( R(T, K_n) \geq (p - 1)(n - 1) + 1 \).

We now show that \( R(T, K_n) \leq (p - 1)(n - 1) + 1 \) for an arbitrary but fixed tree \( T \) of order \( p \geq 2 \) and an integer \( n \geq 2 \). We verify this inequality by induction on \( n \). For \( n = 2 \), we show that \( R(T, K_2) \leq (p - 1)(2 - 1) + 1 = p \). Let there be given a red-blue coloring of \( K_p \). If any edge of \( K_p \) is colored red, then a blue \( K_2 \) is produced. Otherwise, every edge of \( K_p \) is colored red and a red \( T \) is produced. Thus \( R(T, K_2) \leq p \). Therefore, the inequality \( R(T, K_n) \leq (p - 1)(n - 1) + 1 \) holds when \( n = 2 \). Assume for an integer \( k \geq 2 \) that \( R(T, K_k) \leq (p - 1)(k - 1) + 1 \). Consequently, every red-blue coloring of \( K_{(p-1)(k-1)+1} \) contains either a red \( T \) or a blue \( K_k \). We now show that \( R(T, K_{k+1}) \leq (p - 1)k + 1 \). Let there be given a
red-blue coloring of $K_{(p-1)k+1}$. We show that there is either a red tree $T$ or a blue $K_{k+1}$. We consider two cases.

**Case 1.** There exists a vertex $v$ in $K_{(p-1)k+1}$ that is incident with at least $(p-1)(k-1)+1$ blue edges. Suppose that $vv_i$ is a blue edge for $1 \leq i \leq (p-1)(k-1)+1$. Consider the subgraph $H$ induced by the set $\{v_i : 1 \leq i \leq (p-1)(k-1)+1\}$. Thus $H = K_{(p-1)(k-1)+1}$. By the induction hypothesis, $H$ contains either a red $T$ or a blue $K_k$. If $H$ contains a red $T$, so does $K_{(p-1)k+1}$. On the other hand, if $H$ contains a blue $K_k$, then, since $v$ is joined to every vertex of $H$ by a blue edge, there is a blue $K_{k+1}$ in $K_{(p-1)k+1}$.

**Case 2.** Every vertex of $K_{(p-1)k+1}$ is incident with at most $(p-1)(k-1)$ blue edges. So every vertex of $K_{(p-1)k+1}$ is incident with at least $p-1$ red edges. Thus the red subgraph of $K_{(p-1)k+1}$ has minimum degree at least $p-1$. By Theorem 4.11, this red subgraph contains a red $T$. Therefore, $K_{(p-1)k+1}$ contains a red $T$ as well.

Ramsey’s theorem suggests that the Ramsey number $R(F,H)$ of two graphs $F$ and $H$ can be extended to more than two graphs. For $k \geq 2$ graphs $G_1, G_2, \ldots, G_k$, the Ramsey number $R(G_1, G_2, \ldots, G_k)$ is defined as the smallest positive integer $n$ such that if every edge of $K_n$ is colored with one of $k$ given colors, a monochromatic $G_i$ results for some $i$ $(1 \leq i \leq k)$. While the existence of this more general Ramsey number is also a consequence of Ramsey’s theorem, the existence of $R(F,H)$ for every two graphs $F$ and $H$ can also be used to show that $R(G_1, G_2, \ldots, G_k)$ exists for every $k \geq 2$ graphs $G_1, G_2, \ldots, G_k$ (see Exercise 5).

**Theorem 11.8** For every $k \geq 2$ graphs $G_1, G_2, \ldots, G_k$, the Ramsey number $R(G_1, G_2, \ldots, G_k)$ exists.

If $G_i = K_n$, for $1 \leq i \leq k$, then we write $R(G_1, G_2, \ldots, G_k)$ as $R(n_1, n_2, \ldots, n_k)$. When the graphs $G_i$ $(1 \leq i \leq k)$ are all complete graphs of order at least $3$ and $k \geq 3$, only the Ramsey number $R(3,3,3)$ is known. The following is due to Robert E. Greenwood and Andrew M. Gleason [82].

**Theorem 11.9** $R(3,3,3) = 17$.

**Proof.** Let there be given an edge coloring of $K_{17}$ with the three colors red, blue, and green. Since there is no 5-regular graph of order 17, some vertex $v$ of $K_{17}$ must be incident with six edges that are colored the same, say $vv_i$ $(1 \leq i \leq 6)$ are colored green. Let $H = K_6$ be the subgraph induced by $\{v_1, v_2, \ldots, v_6\}$. If any edge of $H$ is colored green, then $K_{17}$ has a green triangle. Thus we may assume that no edge of $H$ is colored green. Hence every edge of $H$ is colored red or blue. Since $H = K_6$ and $R(3,3) = 6$ (Theorem 11.3), it follows that $H$ and $K_{17}$ as well contain either a red triangle or a blue triangle. Therefore, $K_{17}$ contains a monochromatic triangle and so $R(3,3,3) \leq 17$.

To show that $R(3,3,3) = 17$, it remains to show that there is a 3-edge coloring of $K_{16}$ for which there is no monochromatic triangle. In fact, there is an isomorphic factorization of $K_{16}$ into a triangle-free graph that is commonly called the Clebsch graph or the Greenwood-Gleason graph. This graph can be constructed by
beginning with the Petersen graph with vertices \( u_i \) and \( v_i \) \((1 \leq i \leq 5)\), as illustrated in Figure 11.4 by solid vertices and bold edges. We then add six new vertices, namely \( x \) and \( w_i \) \((1 \leq i \leq 5)\). The Clebsch graph \( CG \) (a 5-regular graph of order 16) is constructed as shown in Figure 11.4. This graph has the property that for every vertex \( v \) of \( CG \), the subgraph \( CG - N[v] \) is isomorphic to the Petersen graph.

![Figure 11.4: The Clebsch graph](image)

### 11.2 Turán’s Theorem

Since the Ramsey number \( R(3, 3) = 6 \), it follows that in every red-blue coloring of \( K_n \), \( n \geq 6 \), either a red \( K_3 \) or a blue \( K_3 \) results. We can’t specify which of these monochromatic subgraphs of \( K_n \) will occur, of course, since if too few edges of \( K_n \) are colored red, for example, then there is no guarantee that a red \( K_3 \) will result. How many edges of \( K_n \) must be colored red to be certain that at least one red \( K_3 \) will be produced? This is a consequence of a theorem due to Paul Turán and is a special case of a result that is considered to be the origin of the subject of extremal graph theory.

**Theorem 11.10** Let \( n \geq 3 \). If at least \( \left\lfloor \frac{n^2}{4} \right\rfloor + 1 \) edges of \( K_n \) are colored red, then \( K_n \) contains a red \( K_3 \).