Schemes of modules over gentle algebras and laminations of surfaces.

Based on joint work with
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§1. Reminder: Schemes of modules vs. varieties of modules

Throughout the talk, K will be an algebraically closed field.

Example. Let $Q = \bullet \circlearrowleft \varepsilon$, $I = \langle \varepsilon^2 \rangle \subseteq KQ$, $A = KQ/I$, $\underline{d} = 1$.

$$\operatorname{rep}(A, \underline{d}) := \left\{ M = (M_a)_{a \in Q_1} \in \prod_{a \in Q_1} K^{d_{t(a)} \times d_{s(a)}} \;\middle|\; I \text{ annihilates } M \right\}$$
$$= \{ \alpha \in K \mid \alpha^2 = 0 \} = \{ \alpha \in K \mid \alpha = 0 \} = \{ pt \}$$

$K[\operatorname{rep}(A, \underline{d})] = K[X] / X \cdot K[X] \cong K$ a f.g. reduced K-algebra.

$K[\operatorname{rep}(A, \underline{d})]$ does not quite capture the locally free rank-$\underline{d}$ A-representations with values in an arbitrary commutative K-algebra.

Each legible element $p \in I$, $p \in e_j I e_i$, gives rise to a polynomial function

$$f_p : \prod_{a \in Q_1} K^{d_{t(a)} \times d_{s(a)}} \longrightarrow K^{d_j \times d_i}$$

hence to $d_j d_i$ polynomial functions

$$f_{p,\ell,m} : \prod_{a \in Q_1} K^{d_{t(a)} \times d_{s(a)}} \longrightarrow K$$

$$f_{p,\ell,m} \in K\left[X_{a,u,v} \,\middle|\, a \in Q_1,\ 1 \leq u \leq d_{s(a)},\ 1 \leq v \leq d_{t(a)}\right]$$

$$R(A, \underline{d}) := K\left[X_{a,u,v} \,\middle|\, a \in Q_1,\ 1 \leq u \leq d_{s(a)},\ 1 \leq v \leq d_{t(a)}\right] \Big/ \langle \text{all } f_{p,\ell,m} \rangle$$

<u>Definition.</u> $\underline{\mathrm{rep}}(A, \underline{d}) := \mathrm{Spec}(R(A, \underline{d}))$ (affine) scheme of modules

Example. $Q = \bullet\!\circlearrowleft\!\varepsilon$, $\quad I = \langle \varepsilon^2 \rangle \subseteq KQ$, $\quad A = KQ/I$, $\quad \underline{d} = 1$.

$$R(A,\underline{d}) = K[X]/X^2K[X]$$

$$\underline{\mathrm{rep}}(A,\underline{d}) = \mathrm{Spec}(R(A,\underline{d})) = \{X \cdot R(A,\underline{d})\} = \{pt\}$$ non-reduced scheme

Example. $Q = (1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} 3)$, $\quad I = \langle \alpha\beta \rangle \subseteq KQ$, $A = KQ/I$, $\underline{d} = (2,3,2)$

$$\rho = \alpha\beta \qquad f_\rho : \underbrace{K^{2\times 3}}_{\alpha} \times \underbrace{K^{3\times 2}}_{\beta} \longrightarrow K^{2\times 2}$$

$$(M_\alpha, M_\beta) = \left(\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} \right) \longmapsto \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} \end{bmatrix}$$

$$R(A,\underline{d}) = K[\text{twelve variables}] \big/ \langle \text{four polynomials} \rangle$$

Some properties of $\underline{\mathrm{rep}}(A,\underline{d})$

- For every commutative K-algebra S,
  $\underline{\mathrm{rep}}(A,\underline{d})(S) := \mathrm{Hom}_{K\text{-sch}}(\mathrm{Spec}(S), \underline{\mathrm{rep}}(A,\underline{d})) = \mathrm{rep}_S^{l.f.}(A,\underline{d})$

- In particular, $\underline{\mathrm{rep}}(A,\underline{d})(K) = \mathrm{rep}(A,\underline{d}) = \{\text{closed pts of } \underline{\mathrm{rep}}(A,\underline{d})\}$

- $\mathrm{Spec}(K[\mathrm{rep}(A,\underline{d})]) = \underline{\mathrm{rep}}(A,\underline{d})^{red}$

- For $M \in \mathrm{rep}(A,\underline{d})$: $T_M(\underline{\mathrm{rep}}(A,\underline{d})) / T_M(GL_{\underline{d}}(K)\cdot M) \cong \mathrm{Ext}^1_A(M,M)$
  (Voigt's Lemma)

§2. Block decompositions and generic (τ-)reducedness

Let Q = finite quiver, I = admissible ideal of KQ, $\rho = \{\rho_1, \dots, \rho_m\}$, $\langle \rho \rangle = I$

For $1 \le j \le m$, let $Q(\rho_j)$ be the subquiver consisting of the vertices and arrows involved in ρ_j.

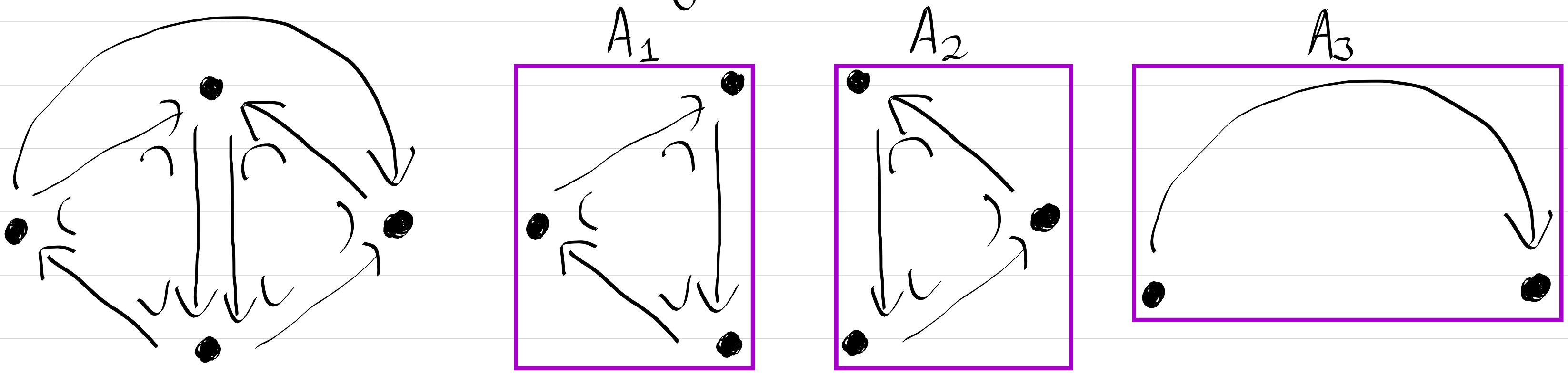
For $a, b \in Q_1$, set $a \sim b$ if they belong to the same $Q(\rho_j)$ for some j.

Let $\sim$ be the equivalence relation generated by this.

Each equivalence class $\rightsquigarrow$ subquiver of Q $\rightsquigarrow$ (non-unital) subalg. of A.

Definition. These (non-unital) subalgebras of A are called ρ-blocks of A

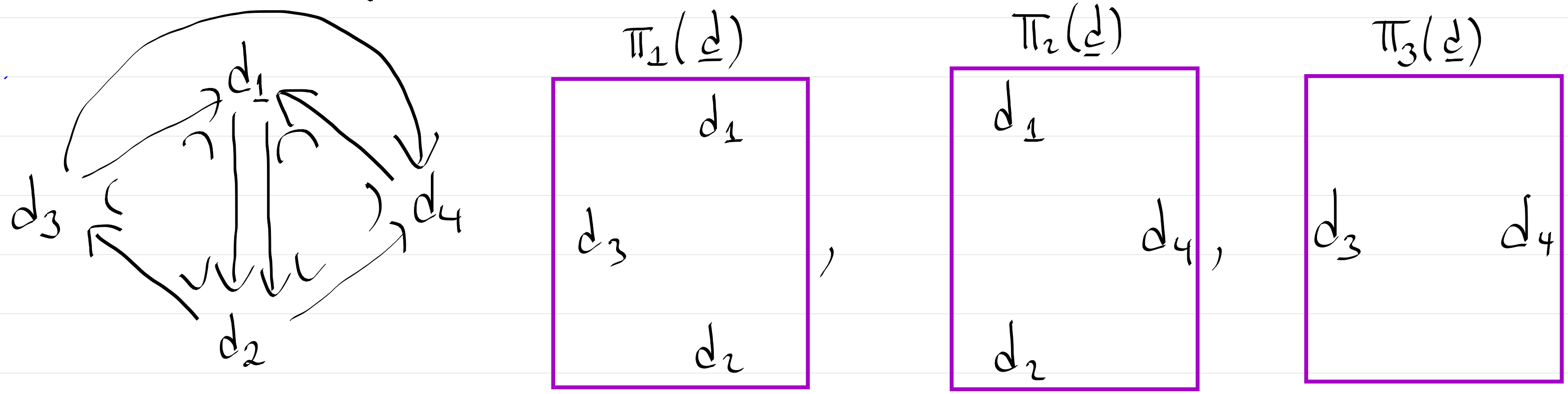
Example.



Let $A_1, \dots, A_t$ be the ρ-blocks of A.

For each dimension vector $\underline{d} \in \mathbb{Z}_{\geq 0}^{Q_0}$ and $1 \leq i \leq t$, let $\pi_i(\underline{d})$ be the corresponding dimension vector for A_t.

Example.



<u>Proposition</u> There are isomorphisms

$\underline{\mathrm{rep}}(A,\underline{d}) \longrightarrow \underline{\mathrm{rep}}(A_1,\pi_1(\underline{d}))\times\dots\times\underline{\mathrm{rep}}(A_t,\pi_t(\underline{d}))$ of schemes

$\mathrm{rep}(A,\underline{d}) \longrightarrow \mathrm{rep}(A_1,\pi_1(\underline{d}))\times\dots\times\mathrm{rep}(A_t,\pi_t(\underline{d}))$ of affine varieties

$\mathrm{Irr}(A,\underline{d}) \longrightarrow \mathrm{Irr}(A_1,\pi_1(\underline{d}))\times\dots\times\mathrm{Irr}(A_t,\pi_t(\underline{d}))$ Irred. comps. of rep(A,−)

$T_M \longrightarrow T_{\pi_1(M)}\times\dots\times T_{\pi_t(M)}$ Tangent spaces of $\underline{\mathrm{rep}}(A,-)$ at $M\in\mathrm{rep}(A,-)$

$T_M^{\mathrm{red}} \longrightarrow T^{\mathrm{red}}_{\pi_1(M)}\times\dots\times T^{\mathrm{red}}_{\pi_t(M)}$ Tangent spaces of $\underline{\mathrm{rep}}(A,-)^{\mathrm{red}}$ at $M\in\mathrm{rep}(A,-)$

Corollary. For $M \in \mathrm{rep}(A,\underline{d})$ and $\mathcal{Z} \in \mathrm{Irr}(A,\underline{d})$:

M is smooth $\iff$ $\pi_i(M)$ is smooth for all $i \in \{1,\dots,t\}$

$\dim T_M = \max\{\dim(\mathcal{Z}) \mid \mathcal{Z} \in \mathrm{Irr}(A,\underline{d}),\ M \in \mathcal{Z}\}$

M is reduced $\iff$ $\pi_i(M)$ is reduced for all $i \in \{1,\dots,t\}$

$\dim T_M = \dim T_M^{red}$

$\mathcal{Z}$ is generically reduced $\iff$ $\pi_i(\mathcal{Z})$ is generically reduced for all i

$\exists\, \mathcal{U} \underset{\text{dense open}}{\subseteq} \mathcal{Z}$, all $M \in \mathcal{U}$ are reduced

Definition. For $M \in \operatorname{rep}(A, \underline{d})$ let

$c_A(M) := \max\{\dim(Z) \mid Z \in \operatorname{Irr}(A, \underline{d}),\ M \in Z\} - \dim(GL_{\underline{d}}(K) \cdot M)$

$e_A(M) := \dim \operatorname{Ext}^1_A(M, M)$

$h_A(M) := \dim \operatorname{Hom}_A(M, \tau_A(M))$

Every $Z \in \operatorname{Irr}(A, \underline{d})$ has a dense open subset on which c_A, e_A, h_A are constant

$\rightsquigarrow$ generic values $c_A(Z)$, $e_A(Z)$, $h_A(Z)$

Voigt's Lemma + Auslander-Reiten formulas $\Rightarrow$ $c_A(Z) \leq e_A(Z) \leq h_A(Z)$

Definition (Geiss-Leclerc-Schröer) $Z \in \operatorname{Irr}(A, \underline{d})$ is generically τ-reduced if $c_A(Z) = e_A(Z) = h_A(Z)$

$\operatorname{Irr}^\tau(A) := \{Z \in \operatorname{Irr}(A) \mid Z \text{ is generically } \tau\text{-reduced}\}$

Theorem (GLFS, De Concini–Strickland 1981 in the acyclic case)
If A is a gentle algebra without loops, then every $Z \in \mathrm{Irr}(A)$ is generically reduced.

Theorem (GLFS) Let A be a gentle Jacobian algebra.
For $Z_1, Z_2 \in \mathrm{Irr}^{\tau}(A)$: $\quad \underline{\dim}(Z_1) = \underline{\dim}(Z_2) \iff Z_1 = Z_2$

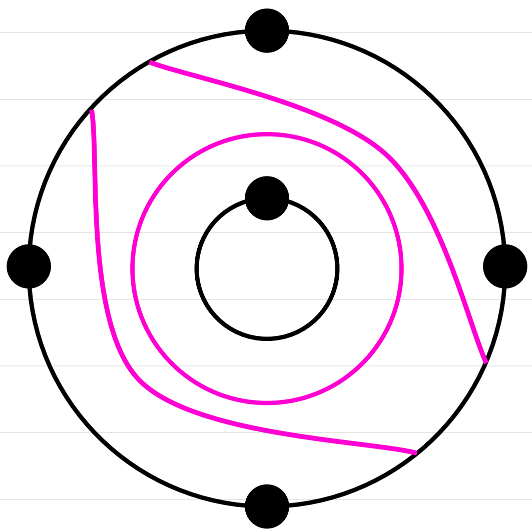
Theorem (GLFS) Let A be a gentle Jacobian algebra and $A_1, \ldots, A_t$ be its ρ-blocks. For $Z \in \mathrm{Irr}(A)$ the following are equivalent:
(i) $Z \in \mathrm{Irr}^{\tau}(A)$
(ii) $\pi_i(Z) \in \mathrm{Irr}^{\tau}(A_i)$ for all $i \in \{1, \ldots, t\}$

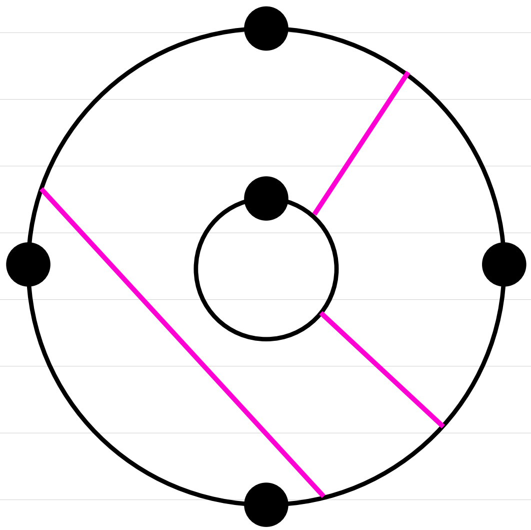
All of this remains valid for (irred. comps. of) decorated representations

§3. Laminations of surfaces and generically τ-reduced components.

We will work with unpunctured surfaces $(\mathbb{S}, \mathbb{M})$

Definition (Fock–Goncharov's $\mathcal{X}$-laminations). A lamination of $(\mathbb{S}, \mathbb{M})$ is a pair $(\gamma, m) = ((\gamma_1, \ldots, \gamma_\ell), (m_1, \ldots, m_\ell))$ where $\gamma_1, \ldots, \gamma_\ell$ are pairwise distinct homotopy classes of curves that do not self-intersect and do not intersect each other, and $m_1, \ldots, m_\ell$ are positive integers, the multiplicities of the laminates $\gamma_1, \ldots, \gamma_\ell$.





Theorem (GLFS) Let T be a triangulation of $(\mathbb{S}, M)$, and let A_T be the associated gentle Jacobian algebra (defined and studied by Assem–Brüstle–Charbonneau–Plamondon and LF). There is a natural bijection $\eta_T : \mathrm{Lam}(\mathbb{S}, M) \longrightarrow \mathrm{decIrr}^{\tau}(A)$

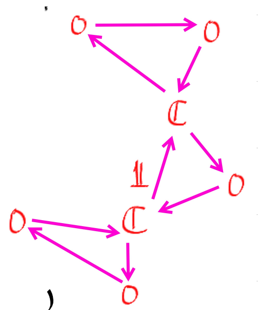
Natural: Additive and the diagram

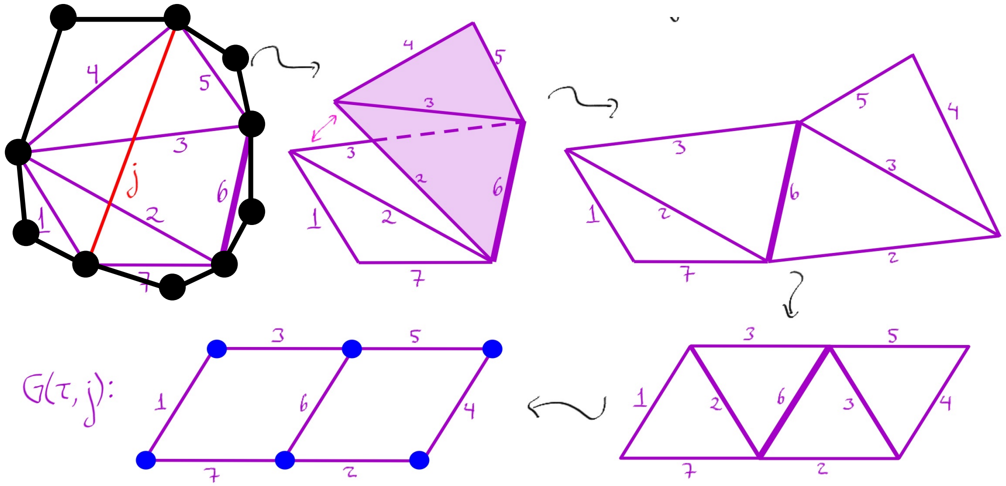
$$\begin{tikzcd}
\mathrm{Lam}(\mathbb{S}, M) \arrow[rr, "\eta_T"] \arrow[dr, "\cong"', "\text{shear coordinates}"'] & & \mathrm{decIrr}^{\tau}(A) \arrow[dl, "\cong", "\text{generic } g\text{-vector}"] \\
& \mathbb{Z}^T &
\end{tikzcd}$$

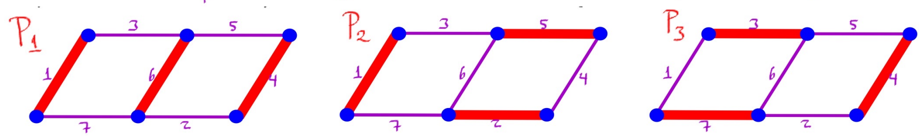
W. Thurston / Fomin–Thurston

Plamondon since $\dim_k(A_T) < \infty$

Theorem (GLFS). In the coefficient-free (upper) cluster algebra $\mathcal{A}(\Sigma, \mathbb{M})$, Geiss-Leclerc-Schröer's set of generic Caldero-Chapoton functions of $A = A_\tau$ is equal to Musiker-Schiffler-Williams' bangle basis.

$M =$ 





$$CC_A(M) := \lambda^{g_A(M)} \cdot F_M(\hat{y}_1, \dots, \hat{y}_7)$$

$$= \frac{\lambda_1\lambda_4\lambda_6 + \lambda_1\lambda_2\lambda_5 + \lambda_3\lambda_4\lambda_7}{\lambda_2\lambda_3}$$

$\lambda(P_1) := \lambda_1\lambda_4\lambda_6$ $\quad\lambda(P_2) := \lambda_1\lambda_2\lambda_5$ $\quad\lambda(P_3) := \lambda_3\lambda_4\lambda_7$

$$\frac{\sum_P \lambda(P)}{\mathrm{mono}(\tau, j)} := \frac{\lambda_1\lambda_4\lambda_6 + \lambda_1\lambda_2\lambda_5 + \lambda_3\lambda_4\lambda_7}{\lambda_2\lambda_3}$$

$$
\begin{tikzcd}
\{\text{cluster monomials in } \mathcal{A}_{(\mathbb{S},\mathbb{M})}\} \arrow[d, leftrightarrow, "1\text{-}1"', "\text{reformulation of [FST]}"] & \underset{\text{[MSW1]}}{\subseteq} & \mathcal{B}_T \arrow[d, leftrightarrow, "1\text{-}1"', "\text{[MSW2] (by def.)}"] \arrow[ddd, bend right=60, equal, "\text{Thm. 1.8}"'] & \\
\{L \in \mathrm{Lam}(\mathbb{S},\mathbb{M}) \mid L \text{ consists of curves}\} \arrow[d, leftrightarrow, "1\text{-}1"', "\text{reformulation of [BZ]+[AIR]}"] & \subseteq & \mathrm{Lam}(\mathbb{S},\mathbb{M}) \arrow[r, leftrightarrow, "1\text{-}1", "\text{[FT]}"'] \arrow[d, leftrightarrow, "1\text{-}1"', "\text{Thm. 1.7}"] & \mathbb{Z}^n \arrow[d, equal] \\
\{Z \in \mathrm{decIrr}^{\tau}(A_T) \mid Z \text{ contains a dense orbit}\} \arrow[d, leftrightarrow, "1\text{-}1"', "\text{reformulation of [DWZ2]+[FST]}"] & \subseteq & \mathrm{decIrr}^{\tau}(A_T) \arrow[r, leftrightarrow, "1\text{-}1", "\text{[P1]}"'] \arrow[d, leftrightarrow, "1\text{-}1"', "\text{[GLS] (by def.)}"] & \mathbb{Z}^n \\
\{\text{cluster monomials in } \mathcal{A}_{(\mathbb{S},\mathbb{M})}\} & \underset{\text{[DWZ2]}}{\subseteq} & \mathcal{G}_T &
\end{tikzcd}
$$

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Some references

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Thank you!