Quivers with potentials associated to triangulated surfaces

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Algebra, Combinatorics and Representation Theory:
in honor of the 60th birthday of Andrei Zelevinsky

Mutations of quivers with potentials

Quiver mutation can be described by means of a three-step procedure on 2-acyclic quivers as follows. Start with a 2-acyclic quiver Q and a vertex k of Q.

(Step 1) For every arrow $\alpha : j \to k$ and every arrow $\beta : k \to i$ in Q, add an arrow $[\beta\alpha] : j \to i$;

(Step 2) replace each arrow γ incident to k with an arrow γ^* going in the opposite direction;

(Step 3) delete 2-cycles one by one (2-cycles may have been created when applying Step 1).

The result is a 2-acyclic quiver $\mu_k(Q)$, called the *mutation of Q with respect to k*.

A *quiver with potential* (*QP* for short) is a pair (Q, S) consisting of a quiver Q and a potential S on Q, that is, a (posibly infinite) linear combination of cycles of Q. In order to lift the notion of mutation to the level of QPs, one needs an algebraic procedure to delete 2-cycles from a (non-necessarily 2-acyclic) QP. Such algebraic procedure is provided by Derksen-Weyman-Zelevinsky's *Splitting Theorem*, which states:

Theorem 1. Every QP (Q, S) is right-equivalent to the direct sum $(Q_{\mathrm{red}}, S_{\mathrm{red}}) \oplus (Q_{\mathrm{triv}}, S_{\mathrm{triv}})$ of a reduced QP $(Q_{\mathrm{red}}, S_{\mathrm{red}})$ and a trivial QP $(Q_{\mathrm{triv}}, S_{\mathrm{triv}})$. Both of these QPs are uniquely determined by (Q, S) up to right-equivalence.

One then keeps the *reduced part* $(Q_{\mathrm{red}}, S_{\mathrm{red}})$ and deletes the 2-cycles that appear in Q_{triv}. 'Unfortunately', the quiver Q_{red} is not necessarily 2-acyclic, its 2-acyclicity depends heavily on the chosen potential S.

Given a 2-acyclic (Q, S) and a vertex k of Q, Derksen-Weyman-Zelevinsky define $\mu_k(Q, S)$, the *mutation of the QP (Q, S) with respect to k*, to be the reduced part of the QP $(\widetilde{\mu}_k(Q), \widetilde{\mu}_k(S))$, where $\widetilde{\mu}_k(Q)$ is the quiver obtained from Q by applying only the first two steps of quiver mutation, and

$$\widetilde{\mu}_k(S) = [S] + \sum_{\overset{\alpha}{\to} k \overset{\beta}{\to}} \beta^*[\beta\alpha]\alpha^*.$$

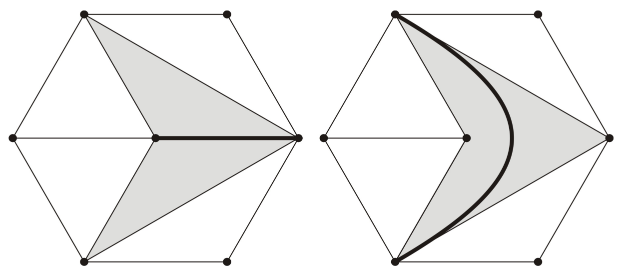
Since reduced parts of QPs are not necessarily 2-acyclic, we see that, 'unfortunately' again, the underlying quiver of the QP $\mu_k(Q, S)$ is not necessarily 2-acyclic, its 2-acyclicity dependes heavily on the chosen potential S. A QP (Q, S) is *non-degenerate* if it is 2-acyclic and every possible sequence of QP-mutations applied to it yields a 2-acyclic QP. Derksen-Weyman-Zelevinsky show that non-degenerate potentials always exist:

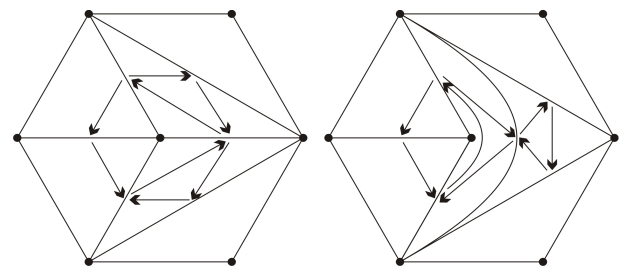
Theorem 2. Every 2-acyclic quiver admits a non-degenerate potential if the ground field is uncountable.

Triangulations of surfaces: their flips and quivers

A *surface with marked points*, or simply a *surface*, is a pair $(\Sigma, \mathbb{M})$, where Σ is a compact connected oriented Riemann surface with (possibly empty) boundary, and $\mathbb{M}$ is a non-empty finite subset of Σ containing at least one point from each connected component of the boundary of Σ.

An *arc on* $(\Sigma, \mathbb{M})$ is a curve on Σ that joins points in $\mathbb{M}$ and is not homotopically trivial. An *ideal triangulation of* $(\Sigma, \mathbb{M})$ is a maximal collection of pairwise non-crossing arcs on $(\Sigma, \mathbb{M})$. A basic operation on ideal triangulations is that of *flip*, the move that replaces a diagonal of any given quadrilateral with the other diagonal. For example, the ideal triangulations of the once-punctured hexagon on the right are related by a flip.



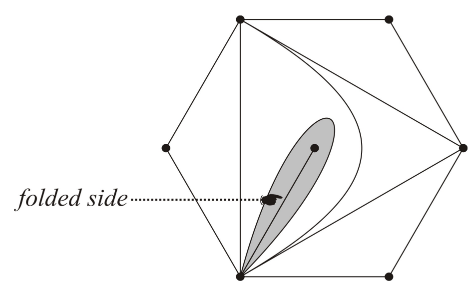


Every ideal triangulation τ has a quiver $Q(\tau)$ associated in a natural way. This was first observed by Fock-Goncharov, Fomin-Shapiro-Thurston and Gekhtman-Shapiro-Vainshtein. The construction of $Q(\tau)$ follows the idea of drawing clockwise-oriented arrows within the triangles of τ, as the figure on the left exemplifies.

Fock-Goncharov, Fomin-Shapiro-Thurston and Gekhtman-Shapiro-Vainshtein realized the following:

Theorem 3. If two ideal triangulations of $(\Sigma, \mathbb{M})$ are related by a flip, then their associated quivers are related by the corresponding quiver mutation.

Many ideal triangulations present the 'unpleasant' feature of having *self-folded triangles*. The *folded side* of such a triangle cannot be flipped within the class of ideal triangulations. In order to be able to flip folded sides, Fomin-Shapiro-Thurston introduced the concept of *tagged triangulation*, a notion more general than that of ideal triangulation, and whose flip combinatorics becomes rather subtle. Fomin-Shapiro-Thurston show that all arcs in a tagged triangulation can be flipped. They furthermore associate a quiver to each tagged triangulation, and prove that Theorem 3 is valid in the more general setting of tagged triangulations.



The quiver with potential of an ideal triangulation

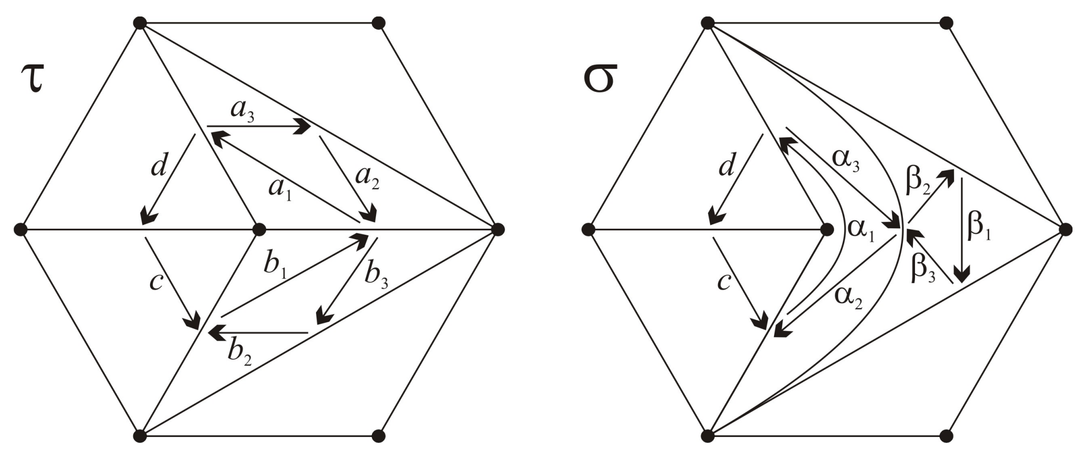
The works of Derksen-Weyman-Zelevinsky and Fomin-Shapiro-Thurston lead to the following natural question:

Question 4. Is it possible to associate to each tagged triangulation τ a potential $S(\tau)$ on the quiver $Q(\tau)$, in such a way that tagged triangulations related by a flip always have QPs related by the corresponding QP-mutation?

A first attempt to answer this question was made in [L1]. Such attempt was not successful at answering the question for all tagged triangulations, but only for ideal triangulations. For an ideal triangulation τ, the definition of the potential $S(\tau)$ is based on the following basic observation: Every '*sufficiently nice*' ideal triangulation τ presents two 'obvious' types of cycles on its quiver $Q(\tau)$, namely:

- the 3-cycles arising from the triangles of τ;
- the cycles running around the punctures of $(\Sigma, \mathbb{M})$.

The potential $S(\tau)$ is then defined as the sum of all these 'obvious' cycles.[1] For example, the potentials associated to the ideal triangulations



are $S(\tau) = a_1a_2a_3 + b_1b_2b_3 + a_1b_1cd$ and $S(\sigma) = \alpha_1\alpha_2\alpha_3 + \beta_1\beta_2\beta_3 + \alpha_1cd$. For ideal triangulations that are not '*sufficiently nice*', the definition of $S(\tau)$ becomes somewhat involved, but we stress the fact that a potential $S(\tau)$ was associated in [L1] to every ideal triangulation τ, even under the presence of self-folded triangles.

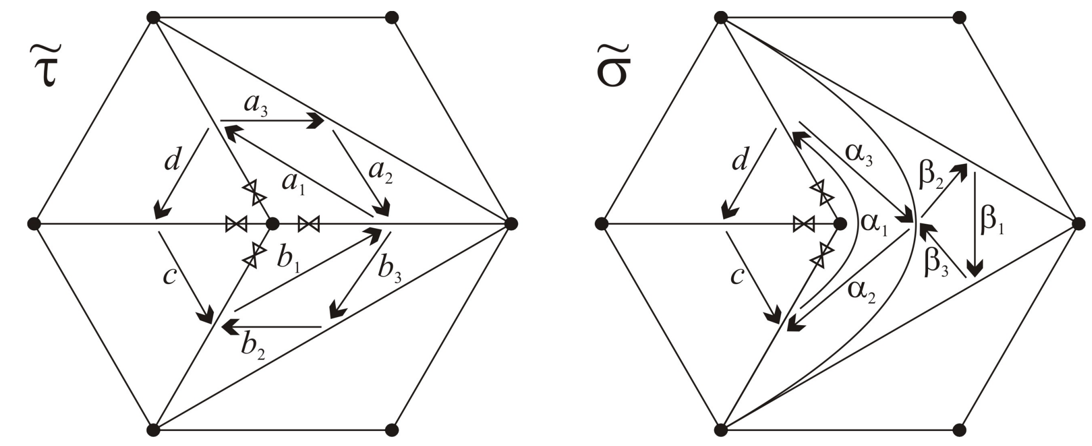
Theorem 5. Let $(\Sigma, \mathbb{M})$ be any surface with marked points. If τ and σ are ideal triangulations of $(\Sigma, \mathbb{M})$ related by the flip of an arc k, then the QPs $(Q(\tau), S(\tau))$ and $(Q(\sigma), S(\sigma))$ are related by QP-mutation with respect to k.

Theorem 5, proved in [L1] by means of a lengthy case-by-case check, is still far from providing an answer to Question 4. Indeed, in the presence of punctures, most tagged triangulations are not ideal. Using a '*deletion of notches*' procedure defined by Fomin-Shapiro-Thurston, it is possible to read an 'obvious' potential on the quiver of any tagged triangulation. What turns out to be hard is not reading such 'obvious' potential, but providing a proof of the '*tagged version*' of Theorem 5. The main difficulty consists in showing that Theorem 5 remains true when we apply a flip that takes us out of the class of ideal triangulations, that is, when we flip the folded side of a self-folded triangle.

[1] In the particular case of unpunctured surfaces (necessarily with non-empty boundary), the potential $S(\tau)$ was found and studied by Assem-Brüstle-Charbonneau-Plamondon independently of [L1].

The quiver with potential of a tagged triangulation

Let us illustrate how '*deletion of notches*' allows us to read potentials on tagged triangulations. Consider the tagged triangulations $\widetilde{\tau}$ and $\widetilde{\sigma}$ shown below.



If we delete all notches from $\widetilde{\tau}$ and $\widetilde{\sigma}$, we obtain the ideal triangulations τ and σ depicted on the left side of this poster. Thus, attaching negative signs to the punctures where at least two notches have been deleted, we define $S(\widetilde{\tau}) = a_1a_2a_3 + b_1b_2b_3 - a_1b_1cd$ and $S(\widetilde{\sigma}) = \alpha_1\alpha_2\alpha_3 + \beta_1\beta_2\beta_3 - \alpha_1cd$.

Theorem 6. Let $(\Sigma, \mathbb{M})$ be any surface with marked points, with the only assumption that $(\Sigma, \mathbb{M})$ is not a sphere with less than six punctures. If τ and σ are tagged triangulations of $(\Sigma, \mathbb{M})$ related by the flip of a tagged arc k, then the QPs $(Q(\tau), S(\tau))$ and $(Q(\sigma), S(\sigma))$ are related by theQP-mutation μ_k.

Theorem 6, recently proved in [L4], provides a positive answer to Question 4. We believe it to be true for the 5-punctured sphere as well. The following is an immediate consequence.

Corollary 7. Let $(\Sigma, \mathbb{M})$ be any surface as in Theorem 6. The QP $(Q(\tau), S(\tau))$ associated to a tagged triangulation τ of $(\Sigma, \mathbb{M})$ is always non-degenerate.

References

- [ABCP] I. Assem, T. Brüstle, G. Charbonneau-Jodoin, P-G. Plamondon. *Gentle algebras arising from surface triangulations.* Algebra & Number Theory Vol. 4 (2010), No. 2, 201-229. arXiv:0903.3347
- [DWZ] H. Derksen, J. Weyman, A. Zelevinsky. *Quivers with potentials and their representations I: Mutations.* Selecta Math. 14 (2008), no. 1, 59119. arXiv:0704.0649
- [FST] S. Fomin, M. Shapiro, D. Thurston. *Cluster algebras and triangulated surfaces, part I: Cluster complexes.* Acta Mathematica 201 (2008), 83-146. arXiv:math.RA/0608367
- [LF1] D. Labardini-Fragoso. *Quivers with potentials associated to triangulated surfaces.* Proc. London Mathematical Society (2009) 98 (3): 797-839. arXiv:0803.1328
- [LF4] D. Labardini-Fragoso. *Quivers with potentials associated to triangulated surfaces, part IV: Removing boundary assumptions.* arXiv:1206.1798

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