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Brian Lehmann: The exceptional sets in Manin's Conjecture ↑

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X smooth (geom. int.) proj. / numb. field F

$\mathcal{L}$ ample adelically metrized line bund. on X

$\Rightarrow$ height function $H_{\mathcal{L}}$ measuring "arithmetic complexity"

For any positive B, set

$$N(X(F), H, B) := \# \{ x \in X(F) \mid H(x) \leq B \}$$

(Northcott): $N(X(F), H, B)$ is finite

Want to understand asymptotics of $N(X(F), H, B)$ as $B \longrightarrow \infty$.

Example: $\mathbb{P}^n_{\mathbb{Q}}$ with std. height fct

$$H(x_0 : x_1 : \cdots : x_n) = \max_{i=0}^{n} |x_i|$$

set of rel. prime integers

N: how many tuples with entries $\leq B$ are rel. prime?

$$N(\mathbb{P}^n(\mathbb{Q}), H, B) \sim \frac{2^n B^{n+1}}{\zeta(n+1)}$$

In general we'll need to remove points:

<u>Def</u>: Suppose $\{\pi_j : Y_j \to X\}_{j=1}^{r}$ collection of gen. finite maps onto their images and admit no rat'l section. Any subset of

$$\bigcup_{j=1}^{r} \pi_j(Y_j(F))$$

is called a <u>thin subset of $X(F)$</u>

Manin's Conjecture

Supp. X smooth proj. $/F$

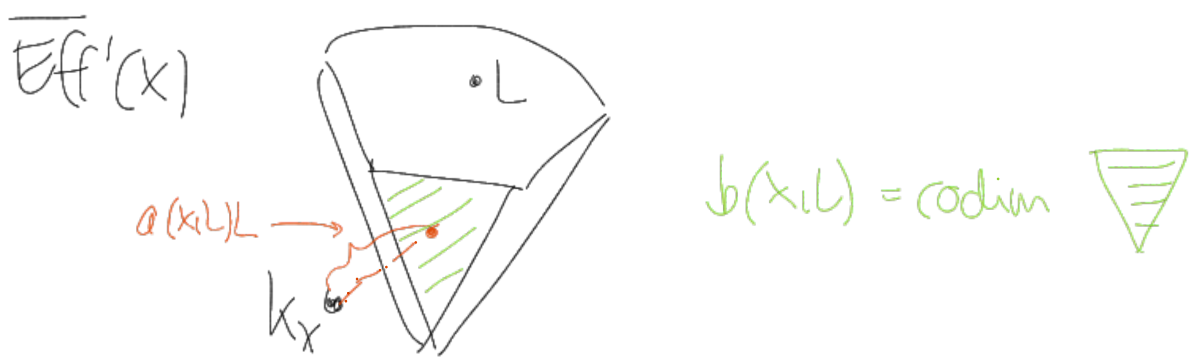
L ample divisor on X

<u>Notation</u>: $\overline{\mathrm{Eff}}^1(X)$ = pseudo-effective cone of divisors

<u>Def</u>: The Fujita invariant is

$$a(X,L) := \inf\{t \in \mathbb{R} \mid [K_X + tL] \in \overline{\mathrm{Eff}}^1(X)\}$$

where K_X = canonical divisor of X



$b(X,L) :=$ codim of minimal face which contains
$K_X + a(X,L)L$

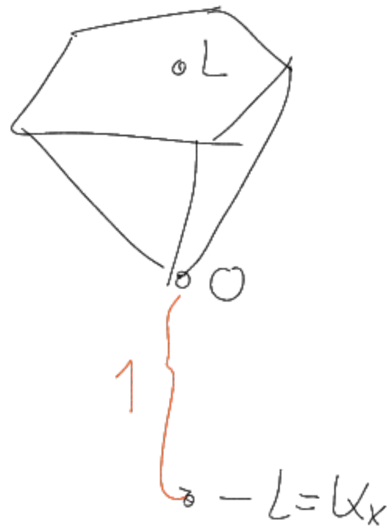
(Turns out to be a Picard rank)

Remark: If X is singular, def. $a(X,L)$ and $b(X,L)$ by pulling back L to a resolution of X.

Example: X Fano, $L = -K_X$

$a(X,L) = 1$ as

$b(X,L) = \rho(X)$



Conjecture: Suppose X smooth Fano / F
$\mathcal{L} \simeq \mathcal{O}_X(L)$ ample adelically metrized
$\exists$ thin subset $Z \subset X(F)$, such that

$$N(X(F) - Z, H, B) \sim c B^{a(X,L)} (\log B)^{b(X,L)-1}$$

where $c = c(F, X, Z, H)$ is Peyre's constant.

Z is called the "exceptional set"

<u>Q</u>: How to determine Z?

<u>Q</u>: Is subtracting thin set sufficient?

Originally conjectured we may choose Z to be non-Zariski-dense. Turns out to be false.

<u>Example</u>: (BT, 1998)

$$X = \{ \sum_{i=0}^{3} x_i y_i^3 = 0 \} \subset \mathbb{P}^3_x \times \mathbb{P}^3_y$$

$$L = -K_X$$

$a(X,L) = 1$ & $b(X,L) = 2$ Picard rank

X admits a fibration by cubic surfaces Y with $a(Y,L) = 1$ & $b(Y,L) \leq 7$

If $\sqrt{-3} \in F$ then a Zariski-dense set of Y has $b > 2$

<u>Example</u>: (LeRudulier) / $\mathbb{Q}$ $\quad X = \mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$, $L = -K_X$

$$\begin{tikzcd} & W \arrow[dl, "Bl_\Delta"'] \arrow[dr, "f"] & \\ (\mathbb{P}^1 \times \mathbb{P}^1)^{\times 2} \arrow[rr, dashed, "2:1"'] & & X \end{tikzcd}$$

$a(X,L) = 1$, $b(X,L) = 3$

$a(W, f^*L) = 1$, $b(W, f^*L) = 4$

pts on $f(W(\mathbb{Q}))$ grows faster than expected.

Conjecture 1: Consider all $f: Y \to X$ that are gen. finite onto image and admit no rat'l section st.

$$\left(a(Y,f^*L), b(Y,f^*L)\right) \geq_{lex} \left(a(X,L), b(X,L)\right)$$

Then: $\bigcup_f f(Y(F))$ is a thin set $\tilde{Z}$

Partial progress towards Conj. 1

Step 1: Prove boundedness over $\bar{F}$ using minimal model program.

Step 2: Prove thinness statement over F using Hilbert's irreducibility theorem.

Example Prove Conj. 1 for $\mathbb{P}^n$; $L = H$ hyperplane

Supp. given $f: Y \to \mathbb{P}^n$ as above

Adjunction theory: if $K_Y + (n+1) f^*H$ is not big then f is birational.

So $a(Y, f^*H) < n+1 = a(\mathbb{P}^n, H)$

So $\tilde{Z} = \emptyset$

Case 1: $Y \subset X$ is a subvariety

Theorem ([HJ]) / $\mathbb{F}$:

Supp X uniruled and L big & nef. Then $\exists$ closed $W \subsetneq X$ such that any subvariety Y with $a(Y, L|_Y) > a(X, L)$ is contained in W.

when $a(Y, L|_Y) = a(X, L)$ but $b(Y, L|_Y) > b(X, L)$, such Y can form a dominant family

Either: • universal family map to X has $\deg \geq 2$

or: • exist monodromy action on Picard groups of fibres

Picard rank ... geom. Picard rank

Theorem: Supp. (X, L) rigid and $\rho(X) = \rho(X_{\bar{F}})$

As we vary over geom. int. subvarieties $Y \subset X$ with

$$(a(Y, L|_Y), b(Y, L|_Y)) >_{lex} (a(X, L), b(X, L))$$

the set $\bigcup_Y Y(F)$ is thin.

Def: (X, L) is rigid, if $\kappa(K_X + a(X, L)L) = 0$.

Rmk: If (X, L) is not rigid, then fibres of canon. fibration have larger a, b-values.

Case 2: $f: Y \to X$ dominant, deg ≥ 2

Easy fact: $a(Y, f^*L) \leq a(X, L)$

Q: When is equality achieved?

Main source: étale in codim 1.

$$\begin{tikzcd} W \arrow[r] \arrow[d] & (\mathbb{P}^1 \times \mathbb{P}^1)^{\times 2} \arrow[d] \\ \mathrm{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1) \arrow[r] & \mathrm{Chow}^2(\mathbb{P}^1 \times \mathbb{P}^1) \end{tikzcd}$$

Conjecture: $/F$ Supp. (X, L) rigid.
Up to birat. equiv. $\exists$ only fin. many dom. generically fin. $f: Y \to X$ with
$$a(Y, f^*L) = a(X, L) \ \& \ (Y, L) \text{ rigid}.$$

Hope: Apply Xu's results.

Theorem $/F$: X smooth, L ample, (X, L) rigid
Supp. $f: Y \to X$ dom., gen. fin. and (Y, f^*L) rigid. As we vary over all twists $f^\sigma: Y^\sigma \to X$ with $\sigma \in \mathrm{Gal}(\bar{F}/F)$

$$(a(Y^\sigma, f^{\sigma *}L), b(Y^\sigma, f^{\sigma *}L)) \geq_{lex} (a(X, L), b(X, L))$$

the set $\bigcup_\sigma f^\sigma(Y^\sigma(F))$ is thin.

<u>Q</u>: Should we allow contributions from

$f : Y \to X$ when

$(a(y, f^*L), b(y, f^*L)) = (a(x, L), b(x, L))$?