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Julien Keller: About J-flow, J- and K-stability

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I Setting / Conjecture

$[\omega]$ Kähler class $\quad \omega_\phi = \omega + \sqrt{-1}\,\partial\bar\partial\phi$

$$\boxed{scal(\omega_\phi) = cst} \qquad cscK$$

topol.

Moment map problem:

S, Ω_S sympl. Lie alg

G Lie group

$$\mu : S \longrightarrow Lie(G)^* \qquad \text{moment map}$$

$$\left(d\langle \mu, \xi \rangle = \Omega_S(\vec{X}_\xi, \dots) \right)$$

Integral of moment map

$$I_\mu : S \times G^{\mathbb{C}} \longrightarrow \mathbb{R} \qquad \text{convex.}$$

$\mathcal{J}$ = almost $\mathbb{C}$-structure compatible with ω

$\circlearrowleft$

$Ham(M, \omega)$

"$\mathrm{Ham}_{\mathbb{C}}(M,\omega) / \mathrm{Ham}(M,\omega)$" $\sim$ $\mathcal{H}_\omega = \{\omega_\phi \in [\omega]\}$

$= \{\phi \mid \omega_\phi \in [\omega]\}$

$$V_\omega(\phi) = K_\omega(\phi)$$

$$= \frac{1}{-V} \int_M \int_0^1 \phi_t \big(\mathrm{scal}(\omega_t) - \underbrace{\mathrm{scal}_0}_{\text{average}}\big) \frac{\omega_t^n}{n!}\, dt$$

ϕ_t path from 0 to ϕ

$$\omega_t = \omega + \sqrt{-1}\, \partial\bar{\partial}\phi_t$$

<u>Conj</u> (Mabuchi, Tian)

$\eta(M) = \{0\}$ no holom. vec. field

$\exists$ cscK ω $\iff$ K_ω proper

"$\|\phi\|$" on $\mathcal{H}_\omega$ Kähler potentials

proper: $\|\phi\| \to +\infty$ then $K_\omega(\phi) \longrightarrow +\infty$

$\Uparrow$

Coercivity: $K_\omega(\phi) \geq C_0$ "$\|\phi\|$" $- C_1$, $C_0 > 0$

Tian proved conj 1 for Fano mnfd, $[\omega] = 2\pi c_1(M) > 0$

PSSW

Berman-Darvas-Lu $\Rightarrow$

Conj. 2 (Y-T-D)

$\exists$ cscK metric in $[\omega]$ $\iff$ K polystability $(M, [\omega])$

$K_M > 0$ true

$-K_M > 0$ true

$\to$ ex. Apten–Süss, R. Dervan

Chen's trick:

$$K_\omega(\phi) = \frac{1}{V}\int \log\left(\frac{\omega_\phi^n}{\omega^n}\right)\omega_\phi^n + \int_0^1\int_M \phi_t\Big(\underbrace{-Ric(\omega)\wedge\omega_t^{n-1}}_{\gamma\omega_t - Ric(\omega)} - \gamma\,\omega_t^n\Big)\,dt$$

$\downarrow$ entropy

$$J_{\chi,\omega} = \int_0^1\int_M \phi_t(\chi\wedge\omega_t^{n-1} - \gamma\,\omega_t^n)\,dt$$

$$\gamma = \frac{\int -Ric(\omega)\wedge\omega^{n-1}}{\int \omega^n}$$ topological const.

χ is Kähler

Critical equation:

$$\boxed{\chi\wedge\omega_\phi^{n-1} = \gamma\,\omega_\phi^n}$$ Moment map prob.

Gradient flow $\quad \frac{\partial\phi_t}{\partial t} = \gamma - \frac{\chi\wedge\omega_t^{n-1}}{\omega_t^n}$ $\leftarrow$ $\exists\ \forall\, t>0$

II Geometric quantization

L_1, L_2 ample line bund. on M

$\begin{array}{cc} L_1 & L_2 \\ \shortparallel & \shortparallel \\ [\omega] & [\chi] \end{array}$ $\qquad M \subset \mathbb{P} H^0(L_1^k) = \mathbb{P}^N \circlearrowleft SL(N+1)$

μ_{FS}

<u>Def</u> : $\mu_\chi(i) = \int_M \mu_{FS}(i)\, \chi \wedge i^* \omega_{FS}^{n-1}$

$\in Lie(SU(N+1))^*$

An embedding i is <u>J-balanced</u> if $\mu_\chi(i) = 0$.

$$\begin{tikzcd} L^k \arrow[d] & \mathcal{O}(1) \arrow[d] \\ M \arrow[r, hook, "i"] & \mathbb{P}^N \end{tikzcd}$$

$i^* h_{FS}$ metric L_1^k $\longrightarrow$ J-balanced metric

<u>Theorem 1</u>

If $\exists$ critical metric ω_ϕ, then $\exists$ sequence of J-balanced metrics (unique) $\omega_k = c_1(i^* h_{FS})$ s.t.

$$\omega_k \xrightarrow[k \to +\infty]{} \omega_\phi$$

grad. flow of $\|\mu_\chi\|^2 \xrightarrow[k \to +\infty]{}$ J-flow

From the alg. pt. of view:

$Y^m \subset \mathbb{P}^N$, $\deg Y = d$,

Z = set of $(N-m-1)$-dim planes that intersect Y
$\|$
$\{f=0\} \subset Gr(N-m, N+1)$

$Y \rightsquigarrow [f_Y] \in \mathbb{P}H^0(Gr, \mathcal{O}(d))$

Chowstable if $[f_Y]$ is GIT stable for $SL(N+1)$

$Y, L > 0$ $\quad Y \subset \mathbb{P}(H^0(Y, L^n))^*$

$Y^m \subset (M, L_1)$

$Y \subset \mathbb{P}(M, H^0(L_1^n))^*$

Y is M-Chowstable

1 param subgroup of $GL(H^0(L_1^R))$ $\quad (\mathcal{M}, \mathcal{L})$

$\updownarrow$

- test configuration: proper flat morph.

$\pi: \mathcal{M} \longrightarrow \mathbb{C}$

$\mathbb{C}^*$ action on $\mathcal{M}$ covering the one on $\mathbb{C}$

- equivariant ample line bundle $\mathcal{L}$

s.t. $(\mathcal{M}_t, \mathcal{L}_t) \sim (M, L_1^R)$ $\quad t \neq 0$

R = exp. of the test config.

We have

$h(k) = h^0(\mathcal{M}_0, \mathcal{L}_0^k) = a_0(R)k^n + a_1(R)k^{n-1} + \dots$

$$\underbrace{\omega(k)}_{\text{weight of the action on } H^0(M_0, \mathcal{L}_0^k)} = b_0(R)k^{n+1} + \cdots$$

Same works with $(Y, L|_Y)$ & obtain

$$\hat{h}(k) = \hat{a}_0(R)k^m + \cdots \qquad \text{Hilbert poly}$$

$$\hat{\omega}(k) = \hat{b}_0(R)k^{m+1} + \cdots$$

Y n-chowstable if

$$\hat{\omega}(Rk) - \omega(R)Rk\,\hat{h}(rk)$$

$$= k^{m+1}\underbrace{(\text{chow weight})}_{>0}$$

<u>Def</u>: (M, L_1) as before & $|L_2|$ linear system.

We say $|L_2|$ is <u>(M, L_1)-chowstable</u> if the "generic Chow weight" of λ 1-param. subgroup is positive.

Asymptotic Chow stability $\to (M, L_1^k)$

<u>Theorem 2</u> L_2 very ample. Then (M, L_1, L_2) admit a J-balanced metric on L_1^k iff $|L_2|$ is (M, L_1^k)-chowstable.

III The leading term of the Chow-weight is called a J-weight

$$J_{L_2}(\mathcal{M}, \mathcal{L}) = \frac{\hat{b}_0 a_0 - b_0 \hat{a}_0}{a_0}$$

Def (Legini, Sz) To be J-stable means that $J_{L_2} > 0$ for all $(\mathcal{M}, \mathcal{L})$ test config. with normal central fibre.

Theorem: $\exists$ critical solution $\Rightarrow$ J-stable.

$$DF(\mathcal{M}, \mathcal{L}) = \frac{b_0 a_1 - b_1 a_0}{a_0}$$

- K-semistability as $DF \geq 0 \quad \forall (\mathcal{M}, \mathcal{L})$
- K-polystab. is K sstab. + DF vanishes iff $(\mathcal{M}, \mathcal{L})$ product test config. $\simeq M \times \mathbb{C}$
- K-stab.: $DF > 0$ except for $(\mathcal{M}, \mathcal{L})$ "almost trivial test configuration"
- def. norm $\|(\mathcal{M}, \mathcal{L})\| = J_L(\mathcal{M}, \mathcal{L})$ on test config.
- uniform K-stable if $DF(\mathcal{M}, \mathcal{L}) > \varepsilon \|(\mathcal{M}, \mathcal{L})\|$

- CM-stability
- (Székelyhidi)

Filtration $\mathcal{F}$ $\quad \bigoplus_{k\geq 0} H^0(M, L^k)$ homog. coord ring

$\mathrm{Rees}(\mathcal{F})$, $\mathcal{F}$ fin-gen. if $\mathrm{Rees}(\mathcal{F})$ is

Test config. $\overset{?}{\longleftrightarrow}$ fin. gen. filtration

$\mathcal{F}$ filtration $\rightsquigarrow$ Δ NO body

Boucksom-Chen: constructed convex function G on Δ

norm $\quad \|\mathcal{F}\|^2 = \int_\Delta (G - \bar{G})^2 \, d\mu$

Theorem 3

To test J-stability it is possible to only consider test configurations obtained by blow-ups of a flag ideal.

Coherent Ideal sheaf $I = I_0 + (t) I_1 + \dots + t^N$ $\quad$ t coord on $\mathbb{C}$

blow up on $M \times \mathbb{C}$

$J(M, \chi) = \text{formula}(\chi_1, \chi_2, E, n, f)$

Corollary 1 :

$\gamma L_1 - L_2$ nef, $\gamma > 0$, L_1 ample. Then

(M, L_1, L_2) is uniform J-stable.

Corollary 2:

(M, L_1, K_M) J-sstable with M Ket. Then

(M, L_1) is uniformly K-stable.

Corollary 3

M Ket, L_1 ample, $\gamma L_1 - K_M$ nef, $\gamma \geq 0$. Then

L_1 is uniform K-stable.