06/02/2017 Banff

Klaus Altmann: The combinatorics of polyhedral divisors

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· toric varieties: $T^n \curvearrowright X^n$ combinatorics

$$N, M \qquad \underbrace{\sigma, \Sigma}_{N_{\mathbb{Q}}}, \ \Delta \subset M_{\mathbb{Q}}$$

lower dim torus action

· $T^k \curvearrowright X^n$ $(k \le n)$

⇝ k-dim combinatorics: $N, M \cong \mathbb{Z}^k$ as above

⇝ n-k geometry: Y^{n-k} semi-proj. var.
(proj. over st. affine)
no torus action
inverse limit of all GIT quot.

M, N lattices

X = affine $\quad \sigma \subset N$ polyhedral cone

$$\boxed{\mathrm{Pol}_+(N,\sigma)} = \{ \Delta \subseteq N_{\mathbb{Q}} \mid \mathrm{tail}(\Delta) = \sigma \} \ni \Delta$$

↳ semigrp wrt. Mink. sum

$$\downarrow$$

$$\mathrm{Pol}(N,\sigma)$$

$$\boxed{\begin{array}{l} Y \xrightarrow{\ n-k\ } \text{normal} \\ \big\downarrow \text{proj} \\ Y_0 \quad \text{affine} \end{array}}$$

- polyhedral divisor $\mathcal{D} = \sum_i \Delta_i \otimes D_i \in \mathrm{Pol}(N,\sigma) \otimes_{\mathbb{Z}}$ CaDiv(Y)
  - $\Delta_i \in \mathrm{Pol}_+$
  - D_i prime (effective)
  - CaDiv(Y): Cart. div.

- $u \in \sigma^\vee \subseteq M_{\mathbb{Q}}$

  u is bounded as fct on all Δ

$$\mathcal{D}(u) := \sum_i \min\langle \Delta_i, u\rangle \cdot D_i \in \mathrm{CaDiv}_{\mathbb{Q}}(Y)$$

$$\xrightarrow{D_i \text{ effective}} \quad \mathcal{D}(u+u') \geq \mathcal{D}(u) + \mathcal{D}(u')$$

$(u \longmapsto \mathcal{D}(u)$ pw lin.$)$

- Def: $\boxed{\mathcal{D}\ \underline{p\text{-divisor}}}$ $:\Longleftrightarrow$ semi ample & big
  - semi ample $:\Leftrightarrow$ $\forall\, u \in \sigma^\vee \cap M$ $\mathcal{D}(u)$ semi ample ($\exists$ multip. base pt free)
  - big $\Updownarrow$ $\forall\, u \in (\mathrm{int}\,\sigma^\vee) \cap M$ $\mathcal{D}(u)$ big

Exp: $N = \mathbb{Z}$, $\sigma = [0, \infty)$

$\mathcal{D} = [1, \infty) \otimes_{\mathbb{Z}} \underset{\text{ample}}{\underset{\cap}{D}} \quad \hat{=} \; D$

cone $(Y, D) =: X$

$\mathcal{D} = \sum \Delta_i \otimes D_i \rightsquigarrow \mathcal{O}_Y(\mathcal{D}) := \bigoplus_{u \in \sigma^\vee \cap M} \mathcal{O}_Y(\mathcal{D}(u))$

sheaf of $\mathcal{O}_Y$-algebras

$\Gamma(Y, \mathcal{O}_Y(\mathcal{D})) =: A$ $\mathbb{C}$-algebra

$$\boxed{\begin{tikzcd} Y & \tilde{X} \arrow[l, "\pi"'] \arrow[d, "r"] \\ & X \end{tikzcd}} \quad \begin{aligned} \tilde{X} &:= \operatorname{Spec} \mathcal{O}_Y(\mathcal{D}) \\ X &:= \operatorname{Spec} A \end{aligned}$$

Y almost quotient of X
$\rightsquigarrow$ need to blow up to $\tilde{X}$

<u>Theorem</u> • $\hat{X}, X$ are normal T-varieties

• X affine T-variety $\Rightarrow (Y, \mathcal{D}, \sigma)$ exist
(also $X = TV(\mathcal{D})$ T-variety of $\mathcal{D}$)

semi ample crucial to get fin. type ($\rightsquigarrow$ variety)

Y "Chow-quotient"

Toric downgrade:

$$0 \to N \hookrightarrow \tilde{N} \xrightarrow{p} \tilde{N}_Y^{a} \to 0$$

find: Δ_a^Y $\qquad \delta \qquad \Sigma := p(\delta)$

$$X = TV(\delta)$$

dual seq: $0 \leftarrow \underset{u}{M} \xleftarrow{\deg} \tilde{M} \leftarrow \tilde{M}_Y^{\nabla_u} \leftarrow 0$

$Y := TV(\Sigma)$ toric var. assoc. to Σ (Δ T-action!)

$\cup$

$D_a = \overline{\mathrm{orb}(a)}$ $\qquad a \in \Sigma(1)$ ray

$\mathcal{D}(u) =$ div. $(TV(\Sigma))$

def. $\quad \Delta_a := p_+^{-1}(a) = p^{-1}(a) \cap \delta$

$$\nabla_u := \deg_+^{-1}(a)$$

Functoriality: $\mathcal{D} : (N, Y, \sigma)$, $\mathcal{D} = \sum \Delta_i \otimes D_i$ $\qquad \mathcal{D}' : (N', Y', \sigma')$, $\mathcal{D}' = \sum \Delta_i' \otimes D_i'$

both p-divisors

Goal: $\quad TV(\mathcal{D}) \xleftarrow{?} TV(\mathcal{D}')$ T-equiv.

- $\psi : N' \to N$ for T-equiv.
- $\varphi : Y' \to Y$

$\Rightarrow \psi_* \mathcal{D}'$ (only change Δ_i')

$\Rightarrow \varphi^* \mathcal{D}$

$\left.\right\}$ $\mathrm{Pol}(N', \sigma') \otimes \mathbb{Q}\mathrm{Div}(Y')$

$\varphi_*\mathcal{D}' \subseteq \varphi^*\mathcal{D} + \mathrm{ppdiv}$ ⇝ for every prime div. polytopes should be cont. in each other

same T-variety as $\mathcal{D}'$

$$\begin{tikzcd} & \mathcal{D}'' \arrow[dl] \arrow[d, "\text{``quasi-isom''}"] \\ \mathcal{D} & \mathcal{D}' \arrow[l] \end{tikzcd}$$

$\mathcal{D} = \sum_i \Delta_i \otimes D_i$

- $\pi =$ degen. bundle with generic fibre $= TV(\sigma)$

$$\begin{tikzcd} Y & \tilde{X} \arrow[l, "\pi"'] \arrow[d, "r"] \\ & X \end{tikzcd}$$

- $D_i \subseteq Y$ ⇝ π is degen. in D_i
- Δ_i tells you how the degen. looks like

Example: $A := \bigoplus_{u \in \sigma^\vee \cap M} \Gamma(Y, \mathcal{D}(u)) = A(\underbrace{TV(\mathcal{D})}_{=X})$ coord. ring

$f \in A_\omega$, $\omega \in \sigma^\vee$

Q: what is A_f?

localization $A_f \cong \mathcal{D}_f := \sum_i \mathrm{face}(\Delta_i, \omega) \otimes D_i\big|_{\text{open subset } Y_f}$

$$\begin{tikzcd} \mathcal{D}_f \arrow[d] \\ \mathcal{D} \end{tikzcd}$$

$$TV(\mathcal{D}') \xhookrightarrow{\text{open}} TV(\mathcal{D}) \Rightarrow \mathcal{D}' \leq \mathcal{D} \text{ face}$$

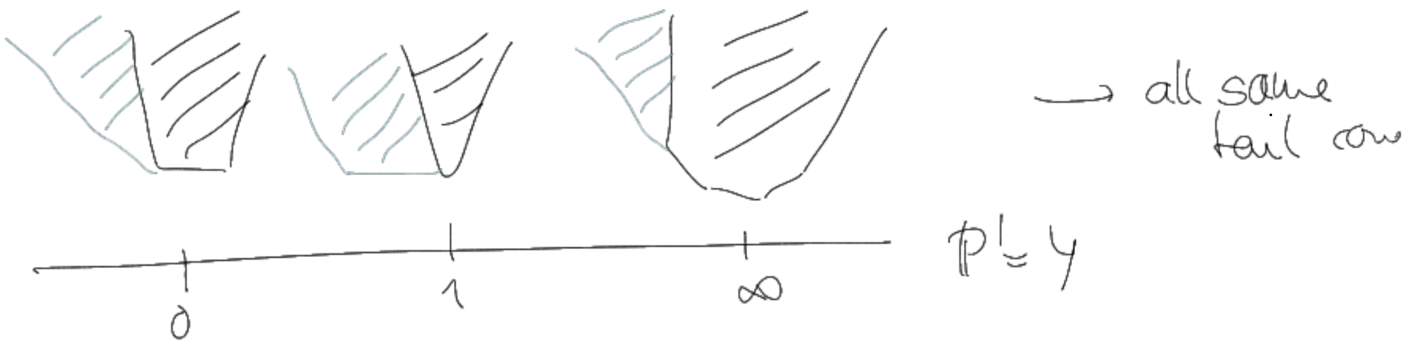
① Complexity 1 $\longleftrightarrow$ Y is a curve (almost true) ($\rightsquigarrow D_i$ are pts) (w. condition)

- $\Delta_i \rightsquigarrow \sum_i \Delta_i =: \deg \mathcal{D}$

  $\deg \mathcal{D}(u) = \min \langle \deg \mathcal{D}, u \rangle$

- semi ample / big $\Leftrightarrow$ $\deg \mathcal{D} \subsetneq \sigma$

↳ understand open embedding above & glue



② Projective T-varieties

toric $\nabla \subseteq M_{\mathbb{Q}}$ lattice polytope

$$\mathbb{P}(\nabla) = TV(\mathcal{N}(\nabla))$$

↳ normal fan

↳ want to generalize

Ilten, Süß: „divisorial polytope" — $(\square, \Psi)$

- $Y = \mathbb{P}^1$
- $\square$ = lattice polytope in $M_{\mathbb{Q}}$

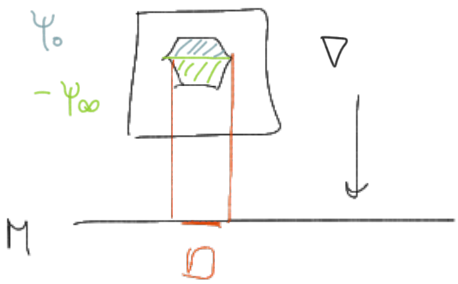
$$\rightsquigarrow \quad \Psi : \square \longrightarrow \mathrm{Div}_{\mathbb{Q}}\, \mathbb{P}^1$$

$$\Psi = \sum_{p \in \mathbb{P}^1} \Psi_p \otimes P$$

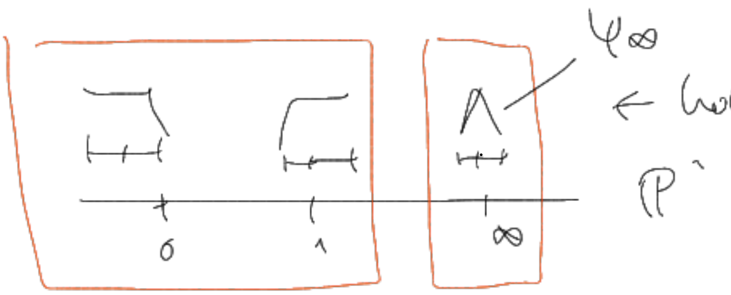
- pw affine, concave
- $u \in \mathrm{int}\,\square \Rightarrow \deg(\Psi(u)) > 0$
- $\forall\, P \in \mathbb{P}^1 = Y \quad \Gamma(\Psi_P)$ has lattice vertices

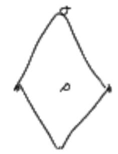
now downgrading:

$$\tilde{M} \twoheadrightarrow M \qquad (\ker = \mathbb{Z})$$



$X = \mathbb{P}(\nabla) \qquad \mathbb{P}^1 \ni 0, \infty$



↳ take sum $= \Lambda = -\Psi_0 \quad \Rightarrow$  toric degen.

Y = proj smooth del Pezzo surface

$\mathrm{Cox}(Y)$ = $\mathrm{Pic}(Y)$-graded $\rightsquigarrow$ T-variety

<u>Q</u>: What is the p-divisor corresp. to $\mathrm{Cox}(Y)$?

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<u>Prop</u>: $\mathcal{D} = \mathrm{id} + \sum_{\substack{E \text{ exep.} \\ \text{curve}}} (\overline{0E} + \mathrm{Nef}(Y)) \otimes E$

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spherical var. $G/H \subset X$

$T \subset B \subset G \curvearrowright X$

take this torus $\rightsquigarrow$ very complex

$X = Gr(2,n) \to$ Kapranov

$N(H)/_H =: T'$ acts on X on $\eta_{Z_{hi}}$

(w. Kiritchenko)

then $Y \supseteq G/N(H)$ spherical var.

has wonderful compactification