Stefano Urbinati: Newton-Okounkov bodies over discrete valuation rings

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1. Motivation
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1. Notation: 
   - $\mathbb{K}$ field of fractions
   - $\mathbb{L}$ residue field
   - $\mathfrak{P}$ uniformizer

Let $\Sigma$ be a polyhedral complex in $\mathbb{R}^n$. We can construct a fan
in $\mathbb{R}^n$:

$\forall P \in \Sigma \Rightarrow$

$\tilde{P} = \{(x, a) \in \mathbb{R}^2 \times \mathbb{R}_+ : \frac{x}{a} \in P\}$

come over $\Sigma$
Note: $\Sigma = \Sigma \cap (R^n \times \{1\})$

Denote $\Sigma$ to be $\Sigma \cap \{R^n \times \{1\}\}$

Let $X(\Sigma)_\mathbb{Z}$ be the toric scheme/\mathbb{Z} associated to $\Sigma$.

The map of fans $\Sigma \to \{0, R_\mathbb{Z}\}$ induces a map $X(\Sigma)_\mathbb{Z} \to \mathbb{A}^2_\mathbb{Z}$. This map is flat and $\mathbb{T}$-equivariant.

\[ \begin{array}{ccc}
\mathbb{Z}[t] & \to & \Theta \\
\pi & \mapsto & \pi
\end{array} \]

\[ \begin{array}{ccc}
X(\Sigma) & \to & X(\Sigma)_\mathbb{Z} \\
\downarrow & & \downarrow \\
\text{Spec}(\Theta) & \to & \mathbb{Z}[t]
\end{array} \]

Mishima Siebert

The general fibre is isomorphic to $X(\Sigma)$.

If $\Sigma$ is integral $\Rightarrow$ special fibre is reduced.

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2. $\Theta$ DVR, $k$ f.d., $k$ res. field.

Let $\mathfrak{X}$ be a semistable scheme over $\Theta$.

$X = \mathfrak{X} \times_{\Theta} \mathbb{Q}$ generic fibre.

Want to construct NO bodies for divisors on $\mathfrak{X}$. 
let \( Y \) be descending flag of proper subschemes
\[ y_i : x = y_0 \supset y_1 \supset \cdots \supset y_{\text{max}} = pt, d = \dim x \downarrow \]
\( v \) valuation assoc. to \( y_i \),
\( v : \quad Y_i \text{ is codim } i \text{ subscheme} \)
\( y_{\text{max}} \text{ smooth pt of each } y_i \)
\( Y_i \text{ is either a semistable scheme over } \mathbb{Q} \text{ or a proper closed of } \mathbb{F}_K \to K \)

**Notation**: the index \( J \) will denote the index s.t. \( Y_{J-1} \text{ semistable over } \mathbb{Q} \) and \( Y_J \) is closed subscheme of central fibre

\( J = 0 \) **TROPICAL CASE** \( \rightarrow \) Newton subdivision
\( J = 1 \) **ARAKELOVAN CASE** \( \rightarrow \) Arithmetic NObodies by King, Yuan

let \( D \) be a divisor on \( x \), flat over \( \mathbb{Q} \)
\[ A_y.(\delta) \text{ defined by sections as in Joaquim talk} \]
\[ \Rightarrow \text{ as in the classical case is convex but} \]
it is **never bounded**.
Lemma: If \( N(t) \in \mathbb{R}^{d+1} \) is viewed as flat on \( \mathbb{R}^d \), then projection along the \( N(t) \) direction is bounded.

Remark: If \( Y_{j+} \) is general \( \Rightarrow \) projection along the \( j \)th component

\[ L \text{ meet central fiber in general pt not special pt} \] (never the case for toric)

\[ \xrightarrow{\text{toric}} \]

**Tropical Case:**

**Theorem:** There is a projection of NO bodies

\[ \text{Pr: } \Delta y_+(J) \rightarrow \Delta y_+(D) \]

on \( \mathbb{R}^d \) on \( X \)

Moreover, \( \Delta y_+(J) \) is given as the overgraph of some convex function

\[ \psi: \Delta y_+(J) \rightarrow \mathbb{R} \]

Remark: When NO body is polyhedral we have a natural induced subdivision of the polytope reconstructing the initial construction.
Remark: For curves it is always polyhedral

3. The case of curves

is directly connected to Baker-Harvey theory of
linear systems on graphs

Note: C semistable curve over Spec(0)

Semistability: C₀ is reduced (central fibre)
and only normal crossings

Def: The dual graph E of C is given by

• \( \mathcal{V}(E) \) set of vertices \( \leftrightarrow \) components in
  the normalization of central fibre \( C₀ \to C \)

• \( \mathcal{E}(E) \) \( \leftrightarrow \) nodes of \( C₀ \)

Notation: \( C₀ \) the component of \( C₀ \) coresp. to a

\[ e = u \circ v \in \mathcal{E}(E) \]

A \underline{divisor} on \( E \) is of the form

\[ D = \sum_{v \in \mathcal{V}(E)} a_v u \circ v \]

\[ \text{over } \mathbb{R} \]

\( D \) is \underline{effective} if \( a_v > 0 \) \( \forall v \)

we study functions \( f: \mathcal{V}(E) \to \mathbb{R} \)
Def. The Laplacian \( \Delta(p) = \sum_{v \in V(\Sigma)} \left( p(v) - \sum_{\omega \in \Sigma} p(\omega) \right) \frac{1}{\operatorname{deg}(\omega)} \) if we denote \( \Sigma_{\Delta} \) as the degree \( \Rightarrow \Delta(p) \) has degree zero.

Link to specialization:
\[
p : \operatorname{Div}(\Sigma) \rightarrow \operatorname{Div}(\Sigma)
\]
\[
\varnothing \quad \longrightarrow \quad \Sigma \quad \sum_{v \in V(\Sigma)} \deg((\Pi^+\varnothing|_{\Sigma})|_{\Sigma})(v)
\]

where if \( \varnothing = \sum p(v)E_v \) is "vertical" then

\[
p(\sum p(v)E_v) = -\Delta(p)
\]

Def. Given a divisor on \( \Sigma \), the linear system is given by

\[
\mathcal{L}(\Lambda) = \{ p : V(\Sigma) \rightarrow \mathbb{R} | \Delta(p) + \Lambda \geq 0 \}
\]

\[
\mathcal{L}^+(\Lambda) = \{ p \in \mathcal{L}(\Lambda) | p(v) \geq 0 \forall v \in V(\Sigma) \}
\]

called the effective linear system.

**Theorem (watermelon tropical case)**

\( y_n \) is a "horizontal" divisor. For \( t \in \mathbb{R} \) denote by

\[
\mathcal{L}_t = \mathcal{L}^+(p(t - t y_n))
\]

Then:

\[\mathcal{L}_t \quad \text{by} \quad (\delta) \quad \text{is the overgraph of} \quad \Theta : \left( \mathbb{R}, \frac{\deg(\delta)}{\deg(y_n)} \right) \rightarrow \mathbb{R} \quad t \rightarrow \omega_{\mathcal{L}_t}(v) \quad \text{maximum of} \quad (\delta)\]
Theorem (Affine General Case) \( Y_t = \mathbb{C}^n \) some component of analytic flow

Let \( L_0 = \left\{ v \in L(\mathbb{R}) \mid \psi(v) = 0 \right\} \) then the Nodality of \( \phi \) is given by points \( b \in \alpha_0(s) = 0 \) and \( \lambda_0(s) = \psi(\mathbb{R})(v) + \max(\Delta(\psi)(s)) \forall \psi \in L_0 \)

Example

Let \( X \) be smooth plane quartic curve of genus \( g(X) = 3 \) this degenerates to

\[ \text{pick a general hyperplane section:} \]
\[ \Psi(\mathbb{R}) = \Lambda = 2(p) + (Q_1) + (Q_2) \]
\[ \Delta(\psi) = (4\psi(p) - 2\psi(Q_1) - 2\psi(Q_2))p \]
\[ + \ldots \quad \text{for } p, Q_1, Q_2 \]

Tropical Case:
\[ Y_t \text{ degree } 1, \; Y_t \in C \quad \text{we}: \begin{cases} \psi \in [0, 2] \quad \psi_t = 0 \\ \psi \in [2, 4] \quad \psi_t = \frac{t - 2}{4} \end{cases} \]