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Stefano Urbinati: Newton-Okounkov bodies over discrete valuation rings

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1. Motivation
2. Definition
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4. Example

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1. Notation: $\mathcal{O}$ DVR

K field of fractions

k residue field

π uniformizer

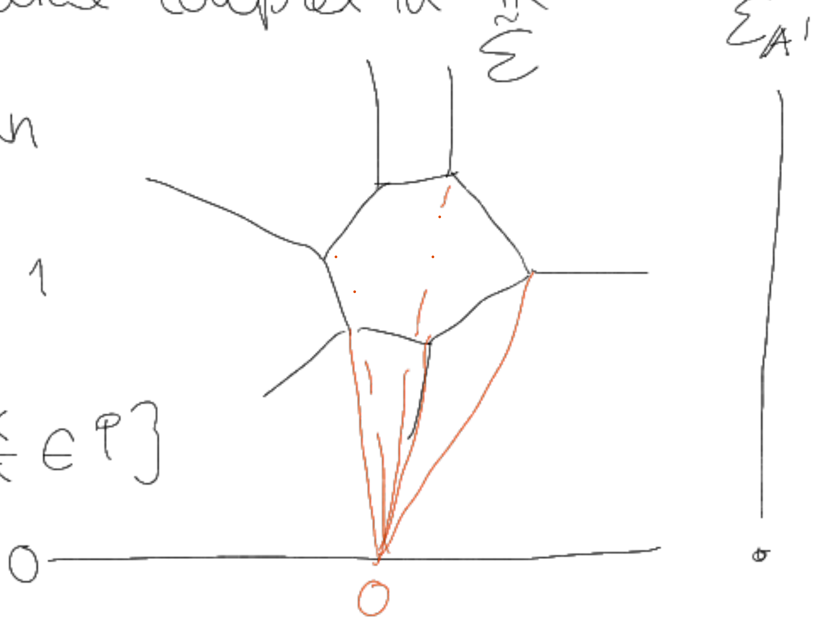
Let Σ be a polyhedral complex in $\mathbb{R}^n$

we can construct a fan in $\mathbb{R}^{n+1}$

$\forall P \in \Sigma \Rightarrow$

$$\tilde{P} = \{ (x,a) \in \mathbb{R}^n \times \mathbb{R}_{\geq 0} \mid \tfrac{x}{a} \in P \}$$

cone over Σ



<u>Note</u>: $\Sigma = \tilde{\Sigma} \cap (\mathbb{R}^n \times \{1\})$

Denote Σ_0 to be $\tilde{\Sigma} \cap \{\mathbb{R}^n \times \{0\}\}$

Let $X(\tilde{\Sigma})_{\mathbb{Z}}$ be the toric scheme$/\mathbb{Z}$ assoc. to $\tilde{\Sigma}$
the map of fans $\tilde{\Sigma} \to \{0, \mathbb{R}_{\geq 0}\}$ induces a map $X(\tilde{\Sigma})_{\mathbb{Z}} \to \mathbb{A}^1_{\mathbb{Z}}$. This map is flat & T-equivariant

$$\mathbb{Z}[t] \to \mathcal{O}, \quad t \mapsto \pi$$

$$\begin{tikzcd} X(\hat{\tilde{\Sigma}}) \arrow[r] \arrow[d] & X(\hat{\tilde{\Sigma}})_{\mathbb{Z}} \arrow[d] \\ i : \mathrm{Spec}(\mathcal{O}) \arrow[r] & \mathbb{Z}[t] \end{tikzcd}$$

<u>Nishimoo-Siebert</u>

the general fibre is isomorphic to $X(\Sigma_0)$
if Σ is integral $\Rightarrow$ Special fibre is reduced

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(2) $\mathcal{O}$ DVR, K FoF, k res. field
Let $\mathfrak{X}$ be a semistable scheme over $\mathcal{O}$
$X = \mathfrak{X} \times_{\mathcal{O}} K$ generic fibre
Want to construct NO bodies for divisors on $\mathfrak{X}$

Let $\mathcal{Y}_\cdot$ be descending flag of proper subschemes

$$\mathcal{Y}_\cdot : \mathcal{X} = Y_0 \supsetneq Y_1 \supset \cdots \supset Y_{d+1} = pt, \; d = \dim X$$

$$\updownarrow$$

$v_\cdot$ valuation assoc. to $\mathcal{Y}_\cdot$

Want: Y_i is codim i subscheme

Y_{d+1} smooth pt of each Y_i

$+$ Y_i is either a semistable scheme over $\mathcal{O}$ or a proper closed of $\mathcal{X} \times_{\mathcal{O}} k$

<u>Notation</u>: the index J will denote the index s.t. Y_{J-1} semistable over $\mathcal{O}$ and Y_J is closed subscheme of central fibre

$\to J = d+1$ TROPICAL CASE $\leftrightarrow$ Newton subdivison

$\to J = 1$ ARAKELOV CASE $\leftrightarrow$ Arithmetic NO bodies by Xing, Yuan

Let $\mathcal{D}$ be a divisor on $\mathcal{X}$, flat over $\mathcal{O}$

$\Delta_{\mathcal{Y}_\cdot}(\mathcal{D})$ defined by sections as in Joaquim's talk

$\rightsquigarrow$ as in the classical case is <u>convex</u> but it is <u>never bounded</u>.

Lemma: If $N(\pi) \in \mathbb{R}^{d+1}$ π viewed as fct on $\mathcal{X}$ then projection along the $N(\pi)$ direction is bounded.

Remark: If Y_{J+} is general $\Rightarrow$ projection along the Jth component

↳ meet central fibre in general pt not special pt (never the case for toric)

TROPICAL CASE:

Thm: There is a surjection of NO bodies

$$P_\pi : \underline{\Delta}_{Y_\cdot}(\mathcal{D}) \longrightarrow \Delta_{Y_\cdot}(D)$$

on $\mathcal{X}$ on X

Moreover, $\underline{\Delta}_{Y_\cdot}(\mathcal{D})$ is given as the over-graph of some convex function

$$\varphi : \Delta_{Y_\cdot}(D) \longrightarrow \mathbb{R}$$

Remark: When NO body is polyhedral we have a natural induced subdivision of the polytope reconstructing the initial construction.

Remark: for curves it is always polyhedral

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③ The case of curves

is directly connected to Baker-Norine theory of linear systems on graphs

Note: $\mathcal{C}$ semistable curve over $\mathrm{Spec}(\mathcal{O})$

Semistability: $\mathcal{C}_0$ is reduced (central fibre) and only normal crossings

Def: The dual graph Σ of $\mathcal{C}$ is given by

- vertices $V(\Sigma)$ set of vertices $\leftrightarrow$ components in the normalization of central fibre $\tilde{C}_0 \to C_0$

- edges $E(\Sigma) \leftrightarrow$ nodes of $\mathcal{C}_0$

Notation: $\mathcal{C}_v$ the component of $\mathcal{C}_0$ corresp. to v

$e = vw \in E(\Sigma)$

A divisor on Σ is of the form $D = \sum_{v \in V(\Sigma)} a_v (v)$

$a_v \in \mathbb{R}$

D is effective if $a_v \geq 0 \quad \forall v$

we study functions $\varphi : V(\Sigma) \longrightarrow \mathbb{R}$

Def: the Laplacian $\Delta(\varphi) = \sum_{v \in V(\Sigma)} \sum_{\substack{e \in E(\Sigma) \\ e = vw}} (\varphi(v) - \varphi(w))\, v$

if we denote $\sum a_v$ as the degree $\Rightarrow \Delta(\varphi)$ has degree zero.

Link to specialization:

$$\rho : \mathrm{Div}(\mathcal{C}) \longrightarrow \mathrm{Div}(\Sigma)$$
$$\mathcal{D} \longmapsto \sum_{v \in V(\Sigma)} \deg(\pi^* \mathcal{O}(\mathcal{D})|_{\mathcal{C}_v})(v)$$

where if $\mathcal{D} = \sum \varphi(v) \mathcal{C}_v$ is "vertical" then
$$\rho\left(\sum \varphi(v) \mathcal{C}_v\right) = -\Delta(\varphi)$$

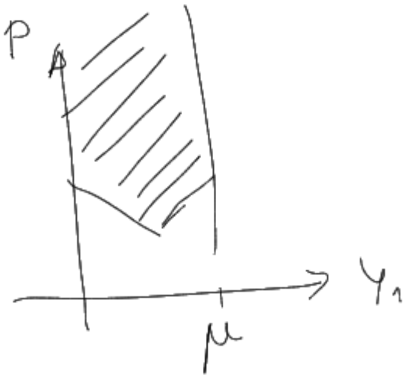
Def: Given Λ divisor on Σ, the linear system is given by
$$L(\Lambda) = \{\varphi : V(\Sigma) \longrightarrow \mathbb{R} \mid \Delta(\varphi) + \Lambda \geq 0\}$$
$$L^+(\Lambda) = \{\varphi \in L(\Lambda) \mid \varphi(v) \geq 0 \ \forall\, v \in V(\Sigma)\}$$

called the effective linear system.

Theorem (Katz, U.) [TROPICAL CASE]

Y_1 is a "horizontal" divisor. For $t \in \mathbb{R}$ denote by $L_t = L^+(\rho(D - tY_1))$. Then:

$\Delta_{Y_1}(\mathcal{D})$ is the overgraph of $\theta : [0, \deg(\mathcal{D})/\deg(Y_1)] \to \mathbb{R}$
$$t \longmapsto \underbrace{\omega_{L_t}(v)}_{\text{minima of } \varphi(v)}$$



<u>Theorem</u> (ARAKELOVIAN CASE) $\quad Y_1 = C_0$ $\begin{cases} \text{some} \\ \text{component of central} \\ \text{fibre} \end{cases}$

Let $L_s = \{ \varphi \in L^+(\rho(D)) \mid \varphi(v) = s \}$ then the NO body of D is given by points btw $a(s) = 0$

and $b(s) = \rho(D)(v) + \max\left(\Delta(\varphi)(s)\big|_{s>0} \ \varphi \in L_s \right)$

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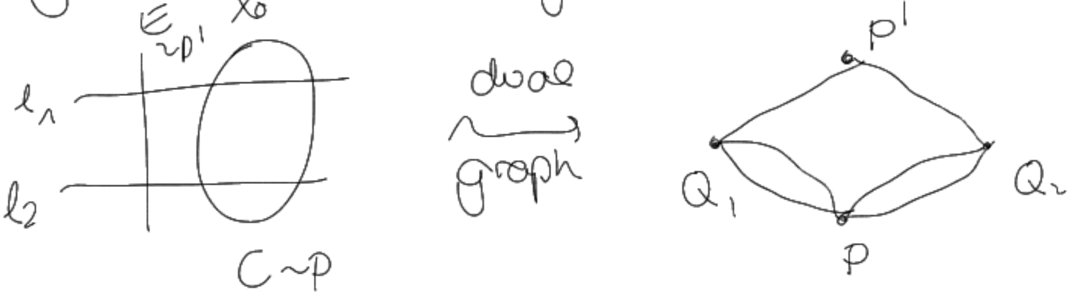
(4) Example

Let X be smooth plane quartic curve of genus $g(X) = 3$. This degenerates to



Q_i corr. to lines

pick D general hyperplane section :

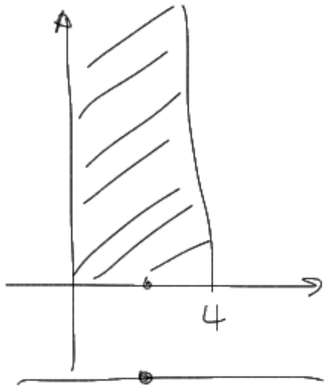
$$\rho(D) = \Delta = 2(P) + (Q_1) + (Q_2)$$

$$\Delta(\varphi) = (4\varphi(P) - 2\varphi(Q_1) - 2\varphi(Q_2))P$$

$$+ \quad \dots \quad \text{for } P', Q_1, Q_2$$

<u>TROPICAL</u> CASE:

Y_1 degree 1, $Y_2 \in C$ $\qquad w_t : \begin{cases} t \in [0,2] & w_t \equiv 0 \\ t \in [2,4] & w_t \equiv \frac{t-2}{4} \end{cases}$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