09/02/2017 Banff

Thomas Eckl: Kähler packings and Newton-Okounkov bodies

X smooth proj. surface $/\mathbb{C}$

$p_1, \dots, p_n \in X$ disjoint pts

L ample line bundle

ω Kähler form on X, $[\omega] = c_1(L) \in H^2(X, \mathbb{Z})$

1. Seshadri & packing constants

Def: $\pi : \tilde{X} \longrightarrow X$ blowup of X in $p_1, \dots, p_n$

$\tilde{X} = Bl_X(p_1, \dots, p_n)$

$E_i = \pi^{-1}(p_i)$ exep. div. for $i = 1, \dots, n$

(n-point) Seshadri constant of X is

$$\varepsilon(X, L; p_1, \dots, p_n) = \sup\left\{ \varepsilon \in \mathbb{Q}_{>0} \,\middle|\, \pi^*L - \varepsilon \sum E_i \text{ is } \mathbb{Q}\text{-ample} \right\}$$

Def: $(B_0^4(r), \omega_{Eucl}) \subset (\mathbb{C}^2, \omega_{Eucl})$

$B_0^4(r) = \{ (z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 < r \}$

$$\omega_{Eucl} = \frac{i}{2\pi} dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2$$

Standard Kähler form (curvature 0)

flat Kähler ball

n-ball Kähler packing constant $r_K(X,\omega;p_1,\dots,p_n)$

$$:= \sup \left\{ r \in \mathbb{R}_{>0} \;\middle|\; \begin{array}{l} \exists \text{ Kähler form } \omega' \in [\omega] \text{ w. hol. embed.} \\ \varphi = (\varphi_1,\dots,\varphi_n) : \coprod_{i=1}^{n} (B_0^4(r), \omega_{Eucl}) \hookrightarrow (X,\omega') \\ \text{s.t. } \varphi_i(0) = p_i \ \& \ \varphi_i^* \omega' = \omega_{Eucl} \end{array} \right\}$$

Def: (n ball) symplectic packing const. $r_S(X,\omega;p_1,\dots,p_n)$

$$:= \sup \left\{ r \in \mathbb{R}_{>0} \;\middle|\; \begin{array}{l} \exists \text{ sympl. embed.} \\ \varphi = (\varphi_1,\dots,\varphi_n) : \coprod_{i=1}^{n} (B_0^4(r), \omega_{Eucl}) \hookrightarrow (X,\omega) \\ \text{s.t. } \varphi_i(0) = p_i \ \& \ \varphi_i^* \omega = \omega_{Eucl} \end{array} \right\}$$

Fact: $r_K(X,\omega;p_1,\dots,p_n) \leq r_S(X,\omega;p_1,\dots,p_n)$

Theorem (Eckl, Witt-Nyström)

$$r_K(X,\omega;p_1,\dots,p_n) = \varepsilon(X,L;p_1,\dots,p_n)$$

Nagata's conjecture:

$X = \mathbb{P}^2_{\mathbb{C}} \supset L$ line, $n \geq 10$. $p_1,\dots,p_n$ in general position

Then $\varepsilon(X,L;p_1,\dots,p_n) = \frac{1}{\sqrt{n}}$ the maximal possible value as $(\pi^*L - \varepsilon \sum E_i)^2$ must be ≥ 0

Theorem (Biran 1997)

$X = \mathbb{CP}^2 \supset L$ line, $\omega = \omega_{FS}$, $n \geq 10$ Then

$$r_S(X, \omega_{FS}, p_1, \dots, p_n) = \frac{1}{\sqrt{n}}$$

2. Explicit symplectic packing

Construct explicitly symplectic embeddings

$$\coprod_{i=1}^{n} (B_0^4(r), \omega_{Eucl}) \hookrightarrow (X, \omega)$$

$X = \mathbb{CP}^2$, $\omega = \omega_{FS}$ Fubini-Study metric

Step 1: Pack a ball instead of $\mathbb{CP}^2$

homog. coord $[z_0 : z_1 : z_2]$

$$(\mathbb{CP}^2, \omega_{FS}) \supset (\mathbb{CP}^2 \setminus \{z_0 = 0\}, \omega_{FS})$$

$$= (\mathbb{C}^2, \omega_{FS})$$

$$\underset{\text{symplectomorph}}{\cong} (B_0^4(1), \omega_{Eucl})$$

$$\frac{(z_1, z_2)}{\sqrt{1 - |z_1|^2 - |z_2|^2}} \longleftarrow\!\mapsto (z_1, z_2)$$

$\hookrightarrow$ but not holom.

$$\omega_{FS} = \frac{i}{2\pi} \partial\bar{\partial} \log(1 + |z_1|^2 + |z_2|^2)$$

<u>Step 2</u>: Pack a prism instead of a ball

$$\square(1) := \{ (x_1, x_2) \in \mathbb{R}^2 \mid 0 < x_1, x_2 < 1 \}$$

$$\Delta(1) := \{ (y_1, y_2) \in \mathbb{R}^2 \mid 0 < y_1, y_2 ,\ 0 < y_1 + y_2 < 1 \}$$

$\square(1) \times \Delta(1) \subset \mathbb{R}^4$ is called <u>prism</u> with standard Euclidean form $\omega_{Eucl} = \frac{1}{\pi}(dx_1 \wedge dy_1 + dx_2 \wedge dy_2)$

$$\varphi : (\square(1) \times \Delta(1), \omega_{Eucl}) \hookrightarrow (B_0^4(1), \omega_{Eucl})$$

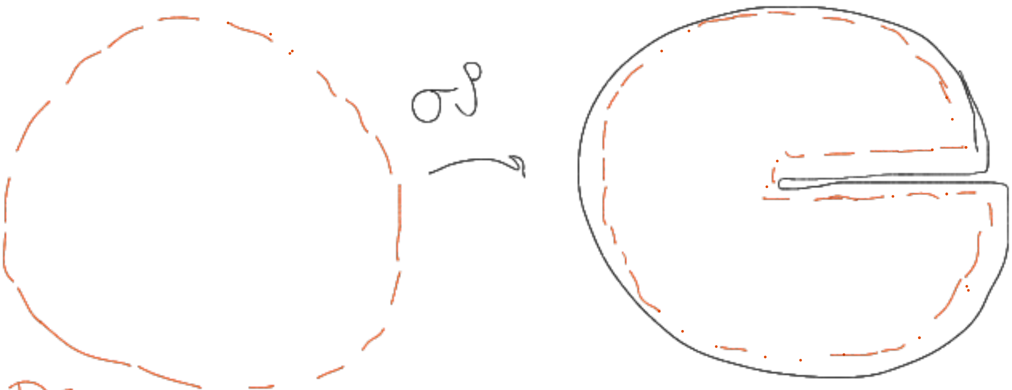
$$(x_1, x_2, y_1, y_2) \longmapsto (\sqrt{y_1}\, e^{2\pi i x_1}, \sqrt{y_2}\, e^{2\pi i x_2})$$

φ is a symplectic embedding.

<u>Step 3</u>: Pack with $\square(1) \times \Delta(r)$ instead of $B_0^4(r)$

slit disk $SD(r) = D(r) - \{x \geq 0, y = 0\}$:

$$\varphi(\square(1) \times \Delta(r)) \subset SD(r) \times SD(r)$$



$D(\rho)$

choose area-preserving diffeo $\sigma^\rho : D(\rho) \hookrightarrow SD(r)$

$\rho < r$ close to r

must be a symplectic embedding

$\Rightarrow \quad B_0^4(\rho) \subset D(\rho) \times D(\rho) \xrightarrow{\sigma^\rho \times \sigma^\rho} SD(r) \times SD(r)$

sympl. embed. wrt ω_{Eucl}

$\psi^\rho(B_0^4(\rho)) \subset \psi(\square(1) \times \Delta(r))$ if σ^ρ is chosen carefully

$\Rightarrow \quad \psi^{-1} \circ \psi^\rho : B_0^4(\rho) \hookrightarrow \square(1) \times \Delta(r)$ sympl. embed. wrt ω_{Eucl} $\boxed{\text{not holomorphic}}$

Step 4: Toric wrapping [Traynor '95]

$$\psi : (\square(1) \times \Delta(1), \omega_{Eucl}) \longrightarrow (B_0^4(1), \omega_{Eucl})$$

extends symplectically to the flat 2-torus instead of $D(1)$

$$\cap$$
$$(\mathbb{T}^2 \times \Delta(1), \omega_{Eucl})$$
by $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$

$$\square(1) \times \Delta(r) \longrightarrow \mathbb{T}^2 \times \Delta(1)$$
$$(x_1, x_2, y_1, y_2) \longmapsto \left((M^T)^{-1}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, M\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \tau \right)$$

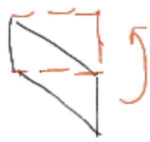
where $M \in Sl_2(\mathbb{R})$ with $\square(1) \xrightarrow{(M^T)^{-1}} \mathbb{R}^2 \longrightarrow \mathbb{T}^2$ injective (holds for $M \in Sl_2(\mathbb{Z})$)

$\tau \in \mathbb{R}^2$, $M(\Delta(r)) + \tau \subset \Delta(1)$

is sympl. embed. wrt ω_{Eucl}.

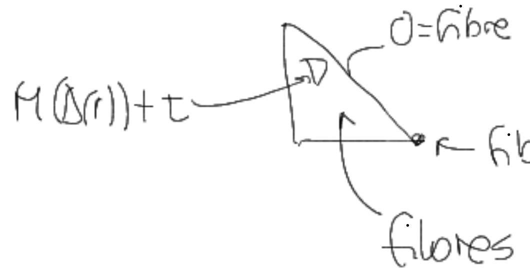
Exp: $M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$: $M(\Delta(r)) = \Delta(r)$

$(M^T)^{-1} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$: $(M^T)^{-1}(\square(1))$

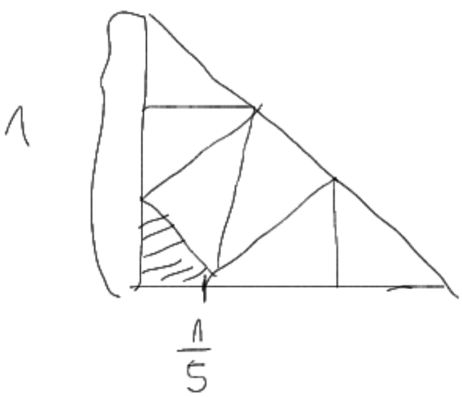


(Partial) vizualization of packing using moment map

$$\begin{array}{ccc} B_0^4(1) & (z_1, z_2) & \text{moment map} \\ \downarrow & \downarrow & \\ & (|z_1|^2, |z_2|^2) & \text{moment polytope} \end{array}$$



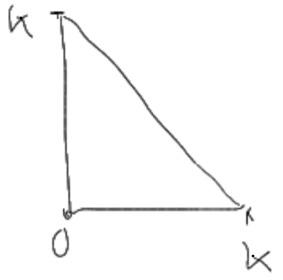
Traynor 6-ball packing



explicit symplectic packing: [Wieck'09]

3. Kähler packing: toric example

$X = \mathbb{P}^2_{\mathbb{C}} \supset L$ line, $p_1 = [0:0:1]$, $p_2 = [0:1:0]$, $p_3 = [0:0:1]$

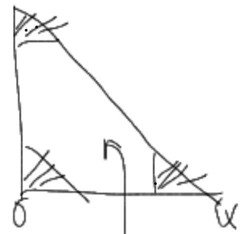


moment polytope ∋ lattice points of kL

$$\Big\updownarrow \; 1:1$$

basis of $H^0(X, \mathcal{O}(kL))$

$$(a,b) \longmapsto x^a y^b z^{k-a-b}$$



moment polytope of $\pi^* kL - l \sum E_i$

<u>Idea</u>: construct Kähler forms ω_δ on X from sections of $\mathcal{O}_{\mathbb{P}^2}(kL)$ s.t. contributions of sections not present in $|\pi^* kL - l \sum E_i|$ vanishes if $\delta \to 0$

$\omega_\delta = \cdots$

$$\varphi_{1,\delta} : B_0^4(R) \hookrightarrow \mathbb{P}^2_{\mathbb{C}}$$

$$(z_1, z_2) \longmapsto [1 : \delta z_1 : \delta z_2]$$

$$\varphi_{1,\delta}^*(\omega_\delta) \xrightarrow{\delta \to 0} \omega_{\text{Eucl}}$$