18/08/16

String cones arising from cluster varieties

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Plan: 0. Motivation
1. Cluster varieties
2. Cluster structure on SL_n/B
3. Connection

References:

[Lit] Littelmann 98

[BZ] Berenstein Zelevinsky 99

[BFZ] —"— Fomin —"— 2003

[FG] Fock Goncharov 2006, 2009

[GHKK] Gross Hacking Keel Kontsevich 2014

[Mag] Magee 2015

0. Motivation

study toric degenerations of flag varieties

Canonical bases for cluster algebras parametrized by superpotential $\forall$ seed [GHKK] $\overset{?}{\longleftrightarrow}$ String cone parametrization of Lusztig's dual can. basis / Kashiwara global basis $\forall$ $\underline{w_0}$ [Lit] [BZ]

$\hookrightarrow$ try to understand [GHKK] by comparing to known [Lit] [BZ]

1. Cluster varieties

[FG] [GHKK]

Let C be a cluster algebra and s seed

$\hookrightarrow$ associate torus T_s

Then mutation $\mu: s \to s'$ yields birat. morph. $\mu: T_s \to T_{s'}$

Def: $\mathcal{A} = \bigcup_{s \text{ seed}} T_s$ glued along mutation is <u>$\mathcal{A}$-cluster variety</u>.

Duality from (cocharacter) lattices of tori induce "dual" torus $\mathcal{T}_s$ $\forall$ seed s with birat. morph. $\tilde{\mu}: \mathcal{T}_s \to \mathcal{T}_{s'}$ $\forall$ mutation $\mu: s \to s'$

Def: $\mathcal{X} = \bigcup_{s \text{ seed}} \mathcal{T}_s$ glued along mutation is <u>$\mathcal{X}$-cluster variety</u> and <u>Fock-Goncharov dual</u> of $\mathcal{A}$.

2. Cluster algebra structure on SL_n/B

$k = \mathbb{C}$, $G = SL_n \supset B$ Borel and $U \subset B$ unipotent rad.

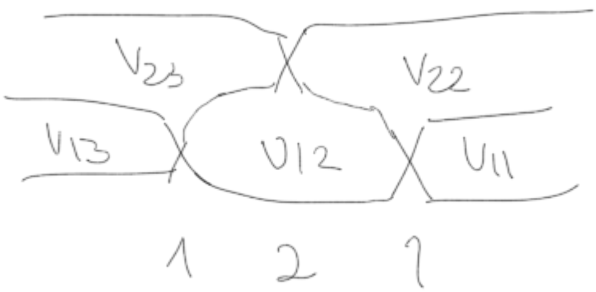
$\qquad\qquad\qquad \cup \; B^- \supset U^-$ opposite Borel

Set $G^{e,w_0} = BeB \cap B^- w_0 B^-$

where $w_0 \in W$ Weyl group is longest elt and e identity.

For every reduced expression $\underline{w_0}$ associate pseudo-line arrangement.

<u>Example</u>: $n = 3$ $\quad \underline{w_0} = s_1 s_2 s_1$



& set of minors

$\{\Delta_3, \Delta_2, \Delta_1, \Delta_{12}, \Delta_{23}\}$

where Δ_I = minor of rows $1, \dots, |I|$ and columns in I.

$\hookrightarrow$ obtain quiver $Q =$

$$\begin{tikzcd}
\boxed{V_{23}} & & \boxed{V_{22}} \arrow[dl] & \\
\boxed{V_{13}} \arrow[r] & V_{12} \arrow[ul] \arrow[rr] & & \boxed{V_{11}}
\end{tikzcd}$$

with $\boxed{}$ frozen vertex.

Now take $\underline{w_0} = s_1 s_2 s_1 s_3 s_2 s_1 \cdots$ with corresp seed as <u>initial seed</u> for cluster algebra $\mathbb{C}[G^{e,w_0}]$

2. Connection

<u>Fact</u> [Mag]

G^{e,w_0} is an $\mathcal{A}$-cluster variety that partially compactifies to SL_n/U (up to codim. 2)

Let $\mathcal{X}$ denote FG-dual of G^{e,w_0}

<u>Theorem</u> [Mag]

There exists $W : \mathcal{X} \to \mathbb{C}$ superpotential s.t. $\{ W|^{trop}_{\mathcal{T}_{s_0}} \geq 0 \}$ for s_0 initial seed, parametrizes a basis of $\mathbb{C}[SL_n/U]$. Moreover, $\{ W|^{trop}_{\mathcal{T}_{s_0}} \geq 0 \}$ is full-dim simplicial cone unimodular equivalent to Gelfand-Tsetlin cone for SL_n.

Remarks:

1) Gelfand-Tsetlin cone is special case of string cone

2) $\forall$ seed s $W|_{\tilde{\mathcal{J}}_s}$ is Laurent polynomial

$\hookrightarrow$ $W|_{\mathcal{J}_s} = \sum_u c_u \underline{x}^u$, $\underline{x}^u = x_1^{u_1} \cdots x_n^{u_n}$ mono.

then $W|^{trop}_{\mathcal{J}_s} = \min_{c_u \neq 0} \{ \sum_{i=1}^n u_i x_i \}$

Theorem [B. Fourier]

Let s be any seed assoc. to reduced expr. $\underline{w_0}$

Then

1. $\{ W|^{trop}_{\mathcal{J}_s} \geq 0 \}$ can be given explicitly

2. $\{ W|^{trop}_{\tilde{\mathcal{J}}_s} \geq 0 \}$ is unimodular equiv. to the string cone assoc. to $\underline{w_0}$.