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The isogeny category of commutative

Commut alg grp / k field

(possibly non aff.)

$\mathcal{C} = \mathcal{C}_k$ = abelian cat (Grothendieck)

$\mathcal{C}/\mathcal{F}$, $\mathcal{F}$ cat. of fin grp schemes :

isogenic category

<u>Definition:</u> An alg group / k is group scheme of finite type.

If char k = 0 such a grp scheme G is a k-variety & smooth [Cartier]

$m : G \times G \longrightarrow G$ mult.

$e \in G(k)$

$i : G \longrightarrow G$ inv.

If char k = p > 0 get schemes

1) $G = \mathbb{G}_a$ simple if char k = 0

$\mathrm{End}(\mathbb{G}_a) = k$ scalar mult.

$= \{p \in k[t] \mid p(x+y) = p(x) + p(y)\}$

If char $k > 0$

$$\mathrm{End}(\mathbb{G}_a) = \left\{ t \mapsto a_0 t + a_1 t^p + \dots + a_m t^{p^m} \right\}_{a_0, \dots, a_m \in k}$$

$$= k\langle F \rangle / (Fa = a^p F,\ a \in k)$$

many fin. subgrp scheme e.g.

$$0 \to \alpha_p \to \mathbb{G}_a \xrightarrow[\text{Frobenius}]{F} \mathbb{G}_a \to 0$$

$$0 \to \underset{\text{(fin. etale)}}{\mathbb{Z}/p\mathbb{Z}} \to \mathbb{G}_a \xrightarrow{F^p - F} \mathbb{G}_a \to 0$$

2) $G = \mathbb{G}_m$ $\quad \mathrm{End}(\mathbb{G}_m) = \mathbb{Z}$, $t \mapsto t^n$, $n \in \mathbb{Z}$

$$0 \to \mu_n \to \mathbb{G}_m \xrightarrow{n} \mathbb{G}_m \to 0$$

nth root of unity (fin. of order n, etale $\Leftarrow$ char $k \nmid n$)

3) E = elliptic curve = smooth proj. curve of genus 1 with k-rat. pt O

$0 \to E[n] \to E \xrightarrow{n} E \to 0$

fin order n^2

$(E, +)$

$$\mathrm{End}(E) = \begin{cases} \mathbb{Z} \\ \text{an order in } \mathbb{Q}[\sqrt{-d}\,] \\ \text{an order in a quaternion algebra over } \mathbb{Q} \end{cases}$$

if E def. over fin. field

All cases occur (Deuring)

(Serre, 1960)

$\mathcal{C}_k$ is artinian, not noeth.

Every object G has a finite filtration

$$0 = G_0 \subset G_1 \subset \dots \subset G_n = G$$

with subquot. G_i/G_{i-1} of one of the types

$\mathbb{G}_a$, torus, abelian var., finite
"simple" "simple"

torus = G s.t. $G_{\bar{k}} \cong \mathbb{G}_{m,\bar{k}} \times \dots \times \mathbb{G}_{m,\bar{k}}$

abelian var. = smooth proj connected alg. group

finite simple = $\overline{\mathbb{Z}/\ell\mathbb{Z}}$ (alg. closure), ℓ prime

α_p, μ_p if $k = \bar{k}$

if $k \neq \bar{k}$ can take Γ Galois grp

$\rho : \Gamma \to GL_n(\mathbb{F}_\ell)$ rep. irred.

$\mathbb{F}_\ell^n$ decends to simple fin. grp scheme

$\mathrm{Hom}(G, H) = 0$ if G, H are elementary of different types except when G is finite

Then $\mathrm{Hom}(G, H)$ is finite unless G is p-torsion

But $\mathrm{End}(\alpha_p) = k = \mathrm{Hom}(\alpha_p, \mathbb{G}_a)$

Theorem:

1960

1) If $k = \bar{k}$, char $k = 0$ then $hd(\mathcal{C}_k) = 1$ (Serre)

(homol. dim.

$\forall G, H \quad \mathrm{Ext}^2_{\mathcal{C}_k}(G, H) = 0$

2) char $k > 0$ then $hd(\mathcal{C}_k) = 2$ (Oort, 66)

3) If k perfect, then $hd(\mathcal{C}_k)$ can be arbitrarily large (Milne, 1970)

Approach: $\mathcal{C}_k$ has not enough injectives, projectives

$$\text{inj} = 0, \quad \text{proj} = \begin{cases} \mathbb{G}_a \times \cdots \times \mathbb{G}_a & \text{char } 0 \\ 0 & \text{char} > 0 \end{cases}$$

Describe all Ext grp btw elem. alg. grps by working in $\hat{\mathcal{C}}$ = (pro-algebraic groups)

(projective limits)

Some examples of extensions

1) $$\mathrm{Ext}^1_{\mathcal{C}}(\mathbb{G}_a, \mathbb{G}_a) = \begin{cases} 0, & \text{if char } k = 0 \\ \text{free mod over } \mathrm{End}(\mathbb{G}_a) \text{ acting on left with gen.} & \end{cases}$$

char > 0

Witt vector

$$0 \to \mathbb{G}_a \xrightarrow{(0,1)} W_2 \xrightarrow{x} \mathbb{G}_a \to 0$$

$W_2 = \mathbb{A}^2$ with $(x, y) + (x', y') = \left(x + x', y + y' + \frac{x^p + x'^p - (x + x')^p}{p}\right)$

2) $\mathrm{Ext}^1_{\mathcal{C}}(A, \mathbb{G}_a) = H^1(A, \mathcal{O}_A) =$ k-vector space of dim = dim A

A — abelian var

3) $\mathrm{Ext}^1_{\mathcal{C}}(G, \mathbb{G}_m) = \mathrm{Pic}(G)^G = \left\{ \begin{array}{c} \text{translation invar. line} \\ \text{bundles on } G \end{array} \right\} / \cong$

G — smooth connected

$0 \to \mathbb{G}_m \to E \to G \to 0$ (Serre, Totaro 2013)

eg. $\mathrm{Ext}^1(\mathbb{G}_a, \mathbb{G}_m) = 0$

$G = A$ abelian var. $\mathrm{Pic}(A) = \mathrm{Pic}^0(A) = \hat{A}(k)$

$\hat{A}$ — dual ab. var

$\mathrm{Ext}^1(A, \mathbb{G}_m) = \hat{A}(k)$

4) If G unipotent, ie $G = \begin{pmatrix} 1 & & * \\ & \ddots & \\ & & 1 \end{pmatrix} = U_n$ for some n

$\mathrm{Ext}^1(G, \mathbb{G}_m)$ and $\mathrm{Pic}(G)$ unknown over k imperfect

(Totaro)

$\mathrm{Ext}^2(\mathbb{G}_a, \mathbb{G}_m)$ unknown

<u>Def</u>: An <u>isogeny</u> is a morph $f: G \to H$, G, H alg. grps s.t. $\mathrm{Ker}(f)$ and $\mathrm{Coker}(f)$ fin.

By inverting all isogenies one defines the <u>isogeny category</u> $\mathcal{C}/\mathcal{F}$

$\mathcal{F}$ = full subcat of fin. grp schemes.

Main result:

1) $\mathcal{C}/\mathcal{F}$ is artinian and noeth.

2) simple objects $\mathbb{G}_a$, simple tori, simple abelian var.

3) $\mathrm{hd}(\mathcal{C}/\mathcal{F}) = 1$ for any field

4) $\mathcal{C}_k \longrightarrow \mathcal{C}_{k'}$ yields $\mathcal{C}_k/\mathcal{F}_k \xrightarrow{\sim} \mathcal{C}_{k'}/\mathcal{F}_{k'}$ (when k'/k purely inseparable)
k'/k field ext.

$\underline{\mathcal{C}}$ full subcat. of $\mathcal{C}/\mathcal{F}$ with obj. smooth & con.

Then $\underline{\mathcal{C}} \xrightarrow{\sim} \mathcal{C}/\mathcal{F}$ equivalence

$$\mathrm{Hom}_{\underline{\mathcal{C}}}(G, H) = \varinjlim_{\substack{H' \subset H \\ \text{finite}}} \mathrm{Hom}_{\underline{\mathcal{C}}}(G, H/H')$$