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Commuting differential operators and Fourier-Mukai transforms

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<u>Example</u>: (Wallenberg 1903)

$$\Lambda \cong \mathbb{Z}^2 \subset \mathbb{C} \qquad \wp(x) = \sum_{\lambda \in \Lambda \setminus \{(0,0)\}} \left(\frac{1}{(x+\lambda)^2} - \frac{1}{\lambda^2} \right)$$

(2) $L = \partial^2 - 2\wp(x)$ Lamé operator

(3) $P = 2\partial^3 - 6\wp'(x)\partial - 3\wp(x)$

$$\Rightarrow [L,P] = 0$$

$$P^2 = 4L^3 + g_2 L + g_3 \qquad g_2, g_3 \in \mathbb{C}$$

$$\psi(x,z) = \frac{\sigma(x-z)}{\sigma(x)\sigma(z)} \exp(\zeta(z)x)$$

genus 1 Baker-Akhiezer function

$$\Rightarrow \quad L_x \circ \psi(x,z) = \wp(z)\psi(x,z)$$

$$P_x \circ \psi(x,z) = \wp'(z)\psi(x,z)$$

Example (Dixmier, 1968)

$$Q = \partial^3 + (x^3 + \lambda), \quad \lambda \in \mathbb{C}$$

(4) $L = Q^2 + 2x$

(6) $P = 2Q^3 + 3(Qx + Qx)$

$$\Rightarrow \quad [L, P] = 0$$

$$P^2 = 4L^3 - \lambda$$

Theorem: Let $\mathbb{C} \neq A \subset \mathcal{D} = \mathbb{C}[x][\partial]$ commut.

(I) A is Noeth. and has Kr dim $A = 1$

$\Rightarrow$ ($X_0 = \mathrm{Spec}(A)$ is an integral curve)

(II)

FofA

$$\begin{array}{ccc} Q(A) & \xrightarrow{val_p} & \mathbb{Z} \\ \cup & & \\ \frac{P}{Q} & \longmapsto & \frac{\mathrm{ord}(Q) - \mathrm{ord}(P)}{r} \end{array}$$

$$r = rk(A) = \gcd(\mathrm{ord}(L) \mid L \in A)$$

It defines a smooth point of a projective curve $X = X_0 \cup \{p\}$

spectral curve of A

Example: Wallenberg $r = 1$

Dixmier $r = 2$

III $F = \mathcal{D}/x\mathcal{D} \xrightarrow{\cong} \mathbb{C}[\partial] \circlearrowleft \mathcal{D} \supset A$

$$\overline{a(x)\partial^n} \longmapsto a(0)\partial^n$$

$\Rightarrow$ F is a fin. gen. torsion-free module of rank $r = \mathrm{rk}(A)$

$$Q(A) \otimes_A F = Q(A)^{\oplus r}$$

History:

Wichever 1977

Burchnall–Chaundy, Baker (1923–1928)

Drinfeld, Mumford, Novikov, ...

Nullstellensatz:

$\{\text{pts of } X_0\} \longleftrightarrow \{\text{Hom. } A \xrightarrow{\chi} \mathbb{C}\}$

Def: $\mathrm{Sol}(A, \chi) = \{ f \in \mathbb{C}[[x]] \mid P \circ f = \chi(P) f \ \forall P \in A \}$

Lemma: I.

$$\begin{tikzcd} A/\ker(\chi) \otimes_A F \arrow[rr, "\eta"] & & \mathrm{Sol}(A,\chi)^* \ni (f \overset{\eta}{\longmapsto} f^{(i)}(0)) \\ & F \arrow[ul, "\pi"] \arrow[ur, "\tilde{\eta}"'] & \\ & \mathbb{C}[\partial] \ni \partial^i \arrow[uur, mapsto] & \end{tikzcd}$$

$F = \mathbb{C}[\partial]$

η is an isom. of $\mathbb{C}$-vector spaces.

II. $\mathrm{Sol}(A,\chi) = \ker(R_\chi)$

$R_\chi = \gcd(P - \chi(P)\cdot 1 \mid P \in A)$

$\cap$

$\mathbb{C}((x))[\partial]$

is Fuchsian.

$(\Rightarrow \dim(\mathrm{Sol}(A,\chi)) = \mathrm{ord}(R_\chi))$

Corollary: $A\text{-mod} \ni F \rightsquigarrow \mathcal{F} \in \mathrm{Coh}(X_0)$

$\mathcal{F}|_\chi \cong \mathrm{Sol}(A,\chi)^*$

Theorem: Let $A \subset \mathcal{D}$ be commut. Then $\exists!$ $\mathcal{F} \in TF(X)$

① $\Gamma(X_0, \mathcal{F}) \cong F$ as $\Gamma(X_0, \mathcal{O}_0) \cong A$-mod.

② $H^i(X, \mathcal{F}) = 0 \quad i = 0, 1$

$(\Rightarrow \mathcal{F}$ is semistable$)$

Theorem (Krichever-correspondence)

$$\{A \subset \mathcal{D} \text{ commut.}\} \xrightarrow{\text{surj}} \left\{ (X, p, \mathcal{F}) \;\middle|\; \begin{array}{l} X \text{ integral proj. curve} \\ p \in X \text{ smooth} \\ \mathcal{F} \in TF(X) : H^*(X, \mathcal{F}) = 0 \end{array} \right\}$$

$\bigcup$

$$\{A \subset \mathcal{D} \mid rk(A) = 1\} \xrightarrow{\text{bij.}} \{(X, p, \mathcal{F}) \mid \mathcal{F} \in \overline{Pic^{g-1}}(X) \setminus \Theta\}$$

Example: $A \subset \mathcal{D} \quad rk(A) = 1 = g(A)$

$\Rightarrow \quad A = \mathbb{C}[L_\alpha, P_\alpha]$

$L_\alpha = \partial^2 - 2\wp(x+\alpha) \qquad \alpha \in (\mathbb{C}/\Lambda) \setminus \{(0,0)\}$

$P_\alpha = \quad \dots \qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad Pic^0(\mathbb{C}/\Lambda) \setminus \{\Theta\}$

Question: what happens when $rk(A) \geq 2$?

$g(A) = 0 \longrightarrow A = \mathbb{C}[P]$

$X = \mathbb{P}^1 \quad \mathcal{F} \cong \mathcal{O}(-1)^{\oplus r}, \; r = ord(P)$

$g_a(A) = 1, \; rk(A) = 2$

$\hookrightarrow$ slides