19/08/16

Tensor multiplicities via upper cluster algebras

Simply con. Lie group

classified by Dynkin diag.

Type A_r $\quad SL_{r+1}$

Cat of reps is semisimple

irr reps param. by dominant weight

$$\lambda = \sum_{i=1}^{r} a_i \omega_i$$

dual to coroots

$$L(\mu) \otimes L(\nu) = \bigoplus_{\lambda} c_{\mu\nu}^{\lambda} L(\lambda)$$

$$c_{\mu\nu}^{\lambda} = \sum_{\substack{\omega \\ \omega'}} \varepsilon(\omega)\varepsilon(\omega') \underbrace{K_A(\omega\lambda + \omega'\mu - \nu)}_{\text{constant}}$$

Weyl grp

2000 Knutson Tao

Hive model for type A

Berenstein Zelevinsky

model for all types

Goal: Generalize hive model

G Lie group
$\cup$
$B^- \supset U^-$

consider $[FG]$

$$K[G]^{U^-} \otimes k[G]^{U^-} \otimes k[G]^{U}$$
$$\bigoplus_\mu L(\mu) \qquad \bigoplus_\nu L(\nu) \qquad \bigoplus_\lambda L(\lambda)^\vee$$

$$= \bigoplus_{\mu,\nu,\lambda} L(\mu) \otimes L(\nu) \otimes L(\lambda)^\vee$$

$$C^{\lambda}_{\mu\nu} \, L(\lambda) \otimes L(\lambda)^\vee$$

upper cl. alg

$$\begin{tikzcd}
\mathrm{GUCA}(\Delta^2_Q) \arrow[rr, leftrightarrow, "{[FG]}"] & & A \\
\mathrm{Rep}(\Delta_{\tilde{Q}}, W_{\tilde{Q}}) \arrow[u, "\text{DWZ cat}"', "\text{gen. CC char.}"] \arrow[rr, "K_0(K^2[J])"] & & \text{lattice points} \arrow[u, "\text{basis}"'] \\
\text{iAR } \Delta_{\tilde{Q}} \arrow[u] \arrow[rr, "\text{stability}"] & & \text{boundary modules} \arrow[u, "\text{describe polytope}"']
\end{tikzcd}$$

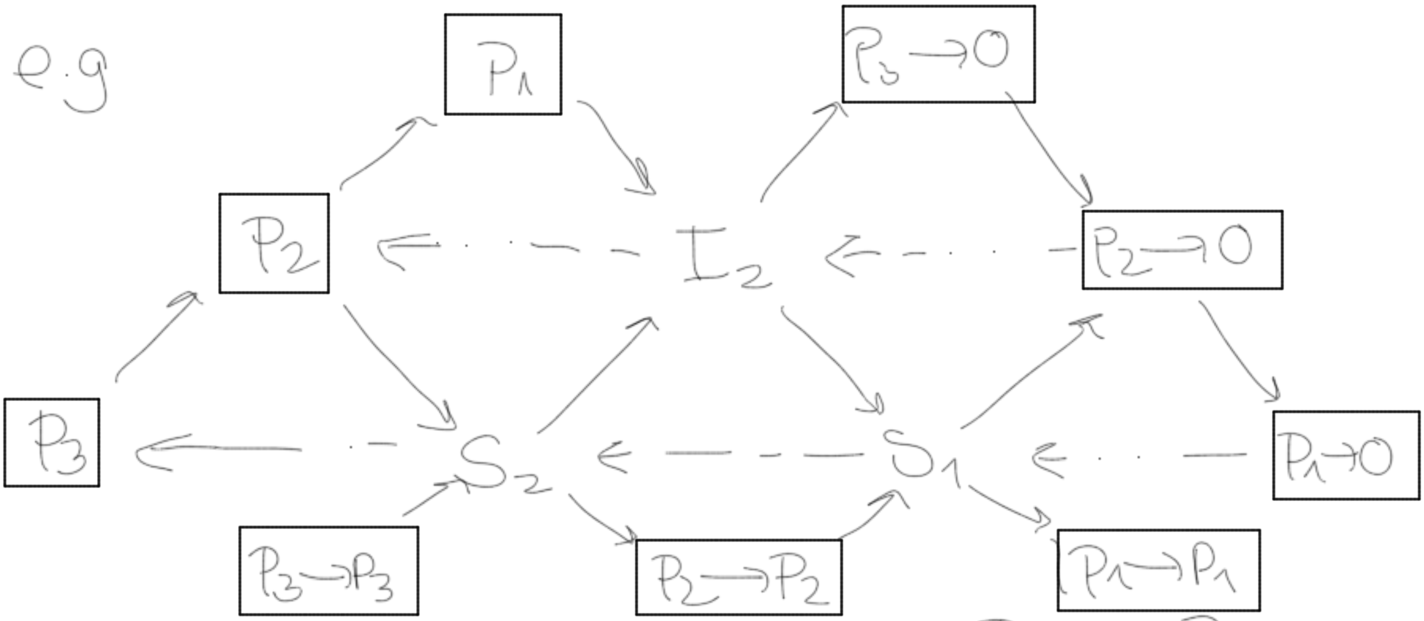
quiver with potential

quiver

poly given by $\begin{cases} x \cdot \sigma = 0 \\ X \cdot H \geq 0 \end{cases}$ $H \to$ matrix

① The quiver $\Delta_{\tilde{Q}}$

e.g.



Def. A presentation is a map $P_+ \longrightarrow P_-$ btw. projectives

also of form $P_i \longrightarrow 0$ called positive

and $P_i \xrightarrow{id} P_i$ called neutral

each row of σ corresp. to $f : P_+ \to P_-$

$$\left(e(f), \quad f^-, \quad f^+ \right)$$

$e(f)$: elementary, zero everywhere but one

$e_2 = (0,1,0 \cdots, 0)$

f^-: mult. of P_- $\subset \mathbb{Z}^r$

f^+: mult. of P_+ $\subset \mathbb{Z}^r$

$r = rk\, G$

How to obtain matrix H?

<u>scline eg.</u>

$$\begin{tikzcd}[column sep=small]
 & & (1,0) \arrow[dl] \arrow[dr] & & (0,3) \arrow[dr] & & \\
 & (2,0) \arrow[dr] & & (1,3) \arrow[ll, dashed] \arrow[ur] \arrow[dr, no head] & & (0,2) \arrow[ll, dashed] \arrow[dr] & \\
(3,0) \arrow[ur] & & (2,3) \arrow[ll, dashed] \arrow[ur, no head] \arrow[dr] & & (1,2) \arrow[ll, dashed] \arrow[ur] \arrow[dr] & & (0,1) \arrow[ll, dashed] \\
 & (3,3) \arrow[ur] & & (2,2) \arrow[ur] & & (1,1) &
\end{tikzcd}$$

each row of H is a dim. vector of a subrep. of a boundary module

e.g. type A all straight lines.

<u>Remark</u>

nonsimply laced: still have AR-quiver

but not all arrows in diag still work

$$\begin{tikzcd}
GLCA \arrow[r, phantom, "\subseteq"] & A \\
\mathrm{Rep}(\Delta_{\mathrm{ilw}\tilde{Q}}) \arrow[u, "\times" marking] \arrow[r] & \text{lattice pts} \arrow[u] \\
\Delta_{\tilde{Q}} \arrow[u] \arrow[r] & \text{bdry modules} \arrow[u]
\end{tikzcd}$$

2. Rep$(\Delta_{\tilde{Q}}, W_{\tilde{Q}})$

construct potential for Δ_Q quiver

$W_Q = \sum$ small triangles

associate Jacobian algebra for (Δ_Q, W_Q)

$\qquad\qquad\qquad\qquad \shortparallel$

$k\Delta_Q/\partial W$ quiver with cyclic relations from W_Q

(keep arrows btw frozen & relations induced from them)

$\qquad\qquad\qquad\qquad\qquad\qquad$ Grothendieck grp

Given a g-vector $\in K^0(K^b(J(\Delta_Q, W_Q)))$

$\qquad\qquad\qquad\qquad\qquad\qquad$ homotopy cat

$g = g^+ - g^-$

$$P(g^+) \xrightarrow{f} P(g^-) \longrightarrow \mathrm{Coker}(g)$$

f generic

Then the cluster character is given by

$$C_W(g) = X^g \sum_e \chi(Gr^e(\mathrm{coker}\, g))\, y^e$$

$\hookrightarrow$ elt in upper cl. alg $\overline{C}(\Delta)$

from B-matrix

g corr. to lattice pts

Summarize:

$$\mathrm{Span}(C_\omega) \subseteq \overset{\text{upper CA}}{\overline{C}(\Delta_{\tilde{Q}}, W_{\tilde{Q}})}$$
$$\subseteq A = (k[G]^{U^-} \otimes k[G]^{U^-} \otimes k[G]^{U^-})^{G}$$

need "$\supseteq$" $\to$ hard to prove

Theorem: equality holds.

Laurent phenomenon:

$$C(\Delta) \subseteq \underset{\substack{(\Delta, X) \sim (\Delta', X') \\ \parallel \\ \text{FZ upper cluster alg}}}{\bigcap} \mathcal{L}_\Delta(X)$$

X cluster
$\mathcal{L}_\Delta(X)$ Laurent poly's in X

slightly chang: require to be polynomial in frozen variables & Laurent-polynomial in mutable variables

g: M - supported g-vectors

$\iff$ $\mathrm{coker}(g)$ not supported on frozens

Poly given by those!

$\rightsquigarrow$ $\mathrm{Hom}(M, \underset{\text{inj}}{I_v}) = 0$ $\Rightarrow$ $\mathrm{Hom}(M\overset{\text{shift}}{[\,]}, I_v) = 0$
"stability cond."

[FG]

$$A = \left(k[G]^{u^-} \times k[G]^{u^-} \times k[G]^{u} \right)^G$$

$$\updownarrow$$

$$[A \times A \times A^\vee / G] \qquad \text{flag}$$

type $\neq A$

up cl. alg $\neq$ cl. alg.

type A unknown