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Srikanth Iyengar:
Local Serre duality for modular representations of finite group schemes

jt D. Benson, H. Krause, J. Pevtsova

I k field Λ fd self inj alg.

stmod Λ = stable module cat of fd Λ-modules

$D(-) = \mathrm{Hom}_k(-,k)$

<u>AR-duality</u>

$D\underline{\mathrm{Hom}}_\Lambda(M,N) \cong \underline{\mathrm{Hom}}_\Lambda(N, \Omega\nu M)$ $M,N \in \mathrm{mod}\,\Lambda$

$\nu(M) = D\Lambda \otimes_\Lambda M$ (Nakayama fctor)

stmod Λ has Serre duality with Serre functor $\Omega\nu$

<u>Notation</u>: $\Lambda = kG$ G fin grp (or sch.)

char $k = p > 0$

<u>eg</u>: $G = (\mathbb{Z}/p\mathbb{Z})^c$ $kG = k[z_1, \ldots, z_c]/(z_1^p, \ldots, z_c^p)$

$\nu(M) = \underbrace{\delta_G \otimes_k M}$ diag G-action

δ_G = Modular char. of G

$\delta_G \cong k \Leftrightarrow kG$ sym. (eg. G fin. grp)

G fin. grp.

AR-duality reads

$$D\,\underline{\mathrm{Hom}}_G(M,N) \cong \underline{\mathrm{Hom}}_G(N,\Omega M)$$

- Tate duality (Cartan-Eilenberg)

For any G there is such a duality locally!

II G any fin. grp (sch.)

$T := \mathrm{StMod}(kG) :=$ stable cat of all kG-modules

- triang. cat with $\Sigma = \Omega^{-1}$
- cpt'ly gen.

$T^c :=$ subcat of cpt obj $= \mathrm{stmod}(kG)$

$M,N \in \mathrm{Mod}\,kG$

$$\underline{\mathrm{Hom}}^*_G(M,N) = \bigoplus_{i\in\mathbb{Z}} \underline{\mathrm{Hom}}_G(M,\Sigma^i N)$$

graded ab. grp

$\underline{\mathrm{End}}^*_G(M)$ graded abelian ring

$R := H^*(G,k) = \mathrm{Ext}^*_G(k,k)$ graded k-alg.

graded-comm. (kG is Hopf alg)

fin gen. as k-alg.

$R = H^*(G,k)$
$T = \mathrm{stMod}\, G$

Point: T is R-linear

For any $M \in \mathrm{Mod}\, G$

$$\phi_M : R \longrightarrow \underline{\mathrm{End}}^*_G(M)$$

$$\begin{array}{ccccc} (k \to \Sigma^i k) & \mapsto & k \otimes_k M & \longrightarrow & \Sigma^i k \otimes M \\ & & \wr\| & & \wr\| \\ & & M & \longrightarrow & \Sigma^i M \end{array}$$

Moreover: $M, N \in \mathrm{Mod}\, G$ the R-actions on

$$\begin{tikzcd} & \underline{\mathrm{Hom}}^*_G(M,N) & \\ \underline{\mathrm{End}}^*_G(N) & & \underline{\mathrm{End}}^*(M) \\ & R \arrow[ul, "\phi_N"] \arrow[ur, "\phi_M"'] & \end{tikzcd}$$

(coincide up to usual sign

$$\phi_N(r) \cdot \xi = (-1)^{|r||\xi|} \xi \, \phi_M(r))$$

A commut. ring $p \in \mathrm{Spec}\, A$, X A-module is

p-local if $X \xrightarrow{\cong} X_p$

p-torsion if $X_q = 0 \ \forall q \notin V(p)$ ie $q \not\supseteq p$

Def: $p \in \mathrm{Proj}\, R$ homog. prime, $p \neq R^{\geq 1}$

$M \in \mathrm{Mod}\, kG$ is <u>p-local</u> if $\forall C \in T^c$ the R-module $\underline{\mathrm{Hom}}^*_G(C,M)$ is p-local.

& is p-local if $\forall C \in T^c$ the R-module $\underline{\mathrm{Hom}}^*_G(C,M)$ is p-torsion.

$\mathcal{T}_p$ = p-local kG-modules

$\Gamma_{\mathcal{V}(p)}(\mathcal{T})$ = p-torsion kG-modules

$$\begin{tikzcd} \Gamma_{\mathcal{V}(p)}(\mathcal{T}) \arrow[r, hook, shift left] & \mathcal{T} \arrow[l, shift left, "\Gamma_{\mathcal{V}(p)}(-)"] \arrow[r, shift left, "(-)_p"] & \mathcal{T}_p \arrow[l, hook, shift left] \end{tikzcd}$$

Set $\Gamma_p(\mathcal{T}) = \Gamma_{\mathcal{V}(p)}(\mathcal{T}) \cap \mathcal{T}_p$ = p-local & p-torsion kG-modules

$\Gamma_p(M) := \Gamma_{\mathcal{V}(p)}(M_p)$ p-local, p-torsion part of M

Fact: $\Gamma_p(\mathcal{T})$ is triang. & cpt'ly gen. & R_p-linear

Theorem (Local Serre duality)

The R_p-linear cat. $(\Gamma_p \mathcal{T})^c$ has Serre duality:

$$\mathrm{Hom}_{R_p}(\underline{\mathrm{Hom}}^*_G(M,N), I(p)) \cong \underline{\mathrm{Hom}}_G(N, \Omega^d \nu(M))$$

inj. hull of R/p

d = Krull-dimension of R/p

1. $\mathcal{T}$ is built out of $\{\Gamma_p \mathcal{T}\}_{p \in \mathrm{Proj} R}$

e.g. $\mathrm{Loc}(M) = \mathrm{Loc}(\Gamma_p M \mid p \in \mathrm{Proj} R)$

$\Gamma_p \mathcal{T}$ are minimal $\otimes$-ideal localizing subcats of $\mathcal{T}$

ie $0 \neq M \in \Gamma_p(\mathcal{T})$ Then

$$\mathrm{Loc}^{\otimes}(M) = \Gamma_p(\mathcal{T})$$

$\rightsquigarrow$ classification of localizing subcats of T

② $\mathfrak{n} \in \operatorname{Proj} R$ closed pt

$$\Gamma_{\mathfrak{m}}(\operatorname{StMod} kG)^c = \text{fin. dim. } kG\text{-mod. with } \Gamma_{\mathcal{V}(\mathfrak{m})} M \cong M.$$

Then local Serre duality for $\Gamma_{\mathfrak{n}}(T)$ = Tate duality

③ K/k field ext. $\quad kG \longrightarrow KG$ hom. of rings

$$H^*(G,k) \xrightarrow{K \otimes_k -} H^*(G,K)$$

$$\begin{aligned} \operatorname{Proj} H^*(G,k) &\longleftarrow \operatorname{Proj} H^*(G,K) \\ \mathfrak{q} \cap H^*(G,k) &\longleftarrow\!\shortmid \quad \mathfrak{q} \end{aligned}$$

$$\operatorname{StMod}(KG) \xrightarrow{\text{restr.}} \operatorname{StMod}(kG)$$

<u>Theorem</u> Fix $\mathfrak{p} \in \operatorname{Proj} R$

$\exists$ (purely transcendental) ext. K/k of degree $\dim(R/\mathfrak{p}) - 1$

and a closed pt $\mathfrak{m} \in \operatorname{Proj} H^*(G,K)$ s.t.

(1) $\mathfrak{m} \cap H^*(G,k) = \mathfrak{p}$

(2) Restriction induces

$$\begin{array}{ccc} \Gamma_{\mathfrak{m}}(\operatorname{StMod} KG) & \supseteq & \Gamma_{\mathfrak{m}}(\cdot\cdot)^c \\ \downarrow & & \downarrow \\ \Gamma_{\mathfrak{p}}(\operatorname{StMod}(kG)) & \supseteq & \Gamma_{\mathfrak{p}}(\cdot\cdot)^c \end{array}$$

full & dense $\quad \square$

Version of Zariski-Weil idea of "generic points"

Corollary

1. Any $M \in \Gamma_p(\mathrm{StMod}\, kG)^c$ is the restriction of a fin. dim. kG-module.

   In particular, M is endofinite in $\Gamma_p(T)$ and also in T.

2. $\Gamma_p(T)^c$ has AR-triangles

Note: ν is periodic on $\mathrm{StMod}\, kG$

Ω is periodic on $(\mathrm{StMod}\, kG)_p$

So $\Omega^{d_p}\nu$ is periodic.

III StMod kG is Gorenstein

Analogy (A, m) commut. local

A Gorenstein, if $\operatorname{inj\,dim} A < \infty$

Thm (Grothendieck) A Gor. $\iff$ $R\Gamma_m A \cong \Sigma^{-\dim A} I(m)$

Back to $\mathrm{StMod}\, kG = T$, $H^*(G,k) = R$

$p \in \operatorname{Proj} R$

$$\operatorname{Hom}_R(\underline{\operatorname{Hom}}^*_G(k,-), I(p)) : \mathrm{StMod}\, kG \longrightarrow \mathrm{Ab}$$

$$\cong \qquad \exists$$

$$\underline{\operatorname{Hom}}_G(-, T_p(k))$$

pure injective

Theorem: $\quad T_p(k) \cong \Omega^d \Gamma_p(\nu k)$, where $d = \dim R/p$

- Serre duality is a formal consequence.
- might be viewed as analogue of
  A Gor. $\Rightarrow$ A_p Gor. $\forall\, p \in \operatorname{Spec} A$

G p-group, p prime

$$\operatorname{StMod} kG \equiv K_{ac}(\operatorname{Inj} kG) \hookrightarrow K(\operatorname{Inj} kG) \equiv D(\underbrace{C^*(BG)}_{\text{commut ring spec.}})$$

Dwyer, Greenlees, I.: $C^*(BG)$ is Gor.

Benson, Greenlees: this property localizes.