

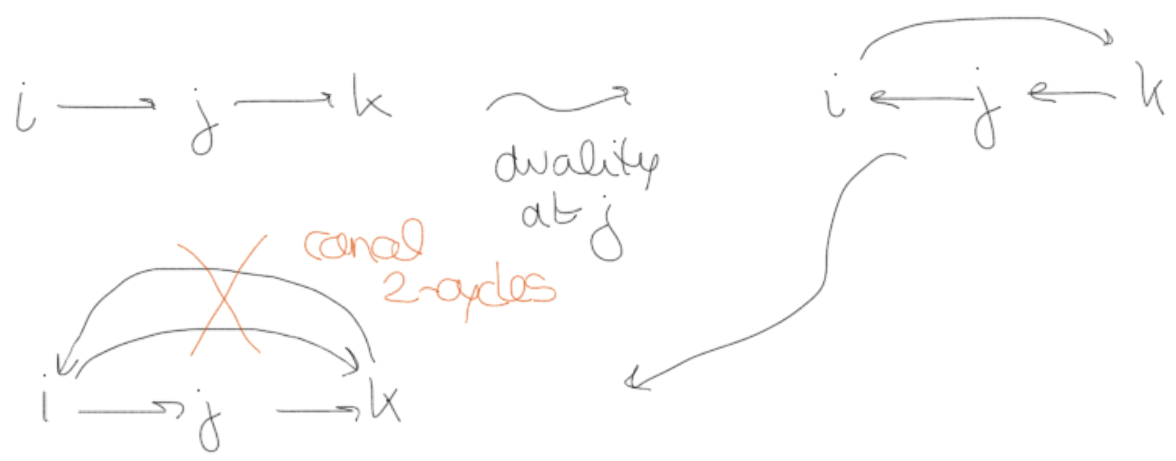
16/08/16

# Kyungyong Lee: Positivity for cluster algebras variables

## History:

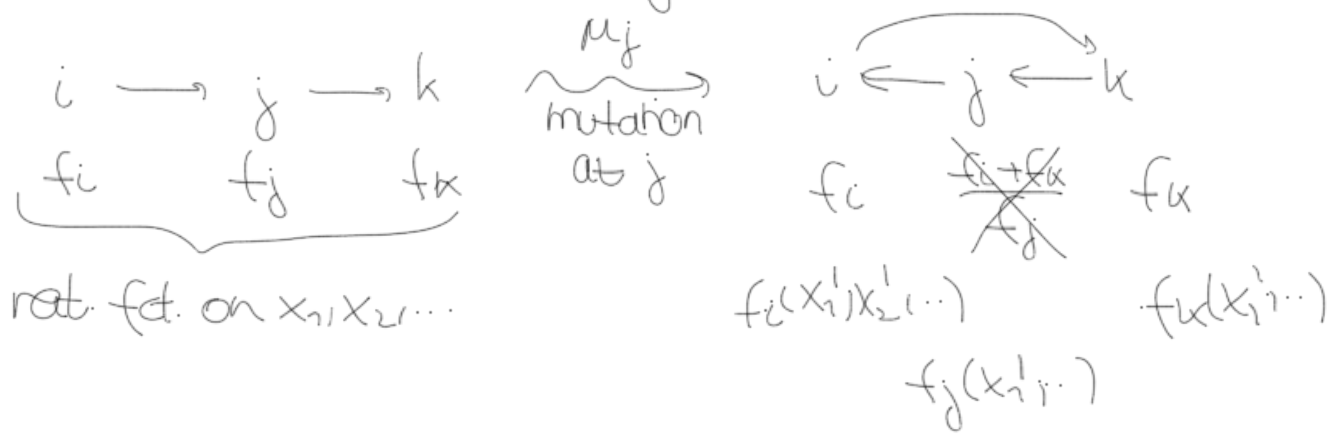
Seiberg duality for quivers (1994)

quiver = directed graph (multiple arrows poss.) w/ loops or 2-cycles



certain moduli of quiver reps are preserved under duality.

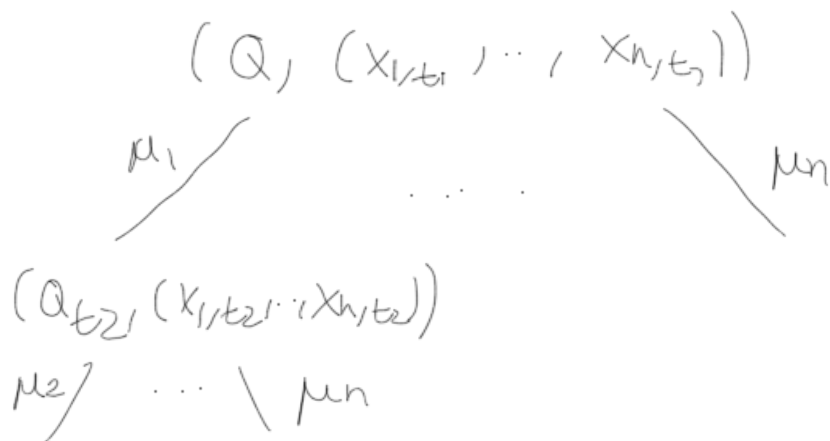
(2000): Fomin and Zelevinsky discovered alg. structure for Seiberg duality.



where

$$x'_p = \begin{cases} x_p & \text{if } p \neq j \\ \frac{x_i + x_k}{x_j} & p = j \end{cases}$$

let  $Q$  be a quiver with  $n$  vertices



Take an arbitrary seq. of mutations from

$$(Q, (x_{1,t_1}, \dots, x_{n,t_n}))$$

$\Rightarrow$  the resulting seed  $(Q', \underbrace{(t_1, \dots, t_n)}_1)$

"cluster variables"



$$(Q, (x_{1,1}, x_{2,1})) \xrightarrow{\mu_1} (Q^{op}, \left( \frac{x_{2,2}^r + 1}{x_{1,2}}, x_{2,2} \right))$$

$\downarrow \mu_2$

$$\xrightarrow{\mu_1} \left( Q, \left( \frac{\left( \frac{x_{1,3}^r + 1}{x_{2,3}} \right)^r + 1}{x_{1,3}}, \frac{x_{1,3}^r + 1}{x_{2,3}} \right) \right)$$

gets complicated very quickly

$$\begin{aligned} \text{If } r=1 \quad & \frac{\left( \frac{\frac{x_{24}+1}{x_{14}} + 1}{x_{24}} \right) + 1}{\frac{x_{24}+1}{x_{14}}} = \frac{(x_{24}+1)(x_{14}+1)}{(x_{24}+1)} \frac{x_{14}}{x_{14}x_{24}} \\ & = \frac{x_{14}+1}{x_{24}} \end{aligned}$$

factorization & cancellation occurs!

Theorem (FZ 2000)

Any cluster variable at any seed  $t$  is a Laurent polynomial of  $x_{1t}, \dots, x_{nt}$  with integer coeff.

Conjecture (FZ 2000)

the coeff. are non-negative.

NOT trivial:  $\frac{1+x^3}{1+x} = 1-x+x^2$

partial answers

acyclic: Nakajima (2009 ~ 2014)

Uimura - Qin (2012)

Surfaces: Musiker - Schiffler - Williams

Theorem (L-Schiffler 2013, Gross-Hacking-Keel-Kontsevich 2014)

The conjecture holds in full generality.

Both proofs  $\left\{ \begin{array}{l} \text{have good control over rank 2} \\ \text{cluster variables} \\ \text{use repeated applications} \\ \text{(induction)} \\ \text{of rank 2 computations} \end{array} \right.$

also, both formulas for rank 2 cluster variables can be generalized to those for bases in rank 2 cluster algebras.

Taking a sequence of mutations. Locally (looks like

$$\dots \underbrace{\mu_j \mu_i \mu_j \mu_i \dots \mu_i \mu_j}_{\text{rank 2}} \overbrace{\mu_k \mu_j \dots \mu_k \mu_j}^{\text{rank 2}}$$

$$(\mathbb{Q}_{t_1}, \dots, X_{t_1}, \dots) \xrightarrow{\mu_e} \dots \xrightarrow[\mu_k]{\mu_j} (\mathbb{Q}_{t_2}, X_{t_2}) \dashrightarrow \dots \dashrightarrow (\mathbb{Q}_{t_6}, X_{t_6})$$

$$\begin{array}{c} | \mu_k \\ (\mathbb{Q}_{t_3}, X_{t_3}) \\ | \mu_k \\ (\mathbb{Q}_{t_5}, X_{t_5}) \\ | \mu_j \\ (\mathbb{Q}_{t_4}, X_{t_4}) \end{array} \quad \begin{array}{c} | \mu_k \\ (\mathbb{Q}_{t_6}, X_{t_6}) \\ | \mu_j \\ \vdots \end{array}$$

$$X_{i,t_2} = \sum_{a_1, b_1 \in \mathbb{N}} f_1 X_{j,t_2}^{-a_1} X_{k,t_2}^{-b_1} + \dots + \sum_{a_3, b_3 \in \mathbb{N}} f_3 X_{j,t_2}^{-a_3} X_{k,t_2}^{-b_3}$$

where  $f_i$  are Laurent poly of  $X_{1,t_2}, \dots, \hat{X}_{j,t_2}, \dots, \hat{X}_{k,t_2}, \dots, X_{n,t_2}$

Induction hypothesis: (on the # of rank 2 subseq.)

$f_3$  can be factored into

$$\underbrace{g_3}_{\substack{\text{pos.} \\ \text{Laurent}}} \left( \prod_{a \rightarrow k} x_{a,t_2} + \prod_{b \leftarrow k} x_{b,t_2} \right)^{b_3}$$

$$\Rightarrow f_3 x_{j,t_2}^{a_3} x_{k,t_2} = g_3 \underbrace{x_{j,t_3}^{a_3} x_{k,t_3}^{b_3}}_{\text{pos. along } \mu_j \mu_k}$$

After  $\mu_i \mu_j \dots \mu_k (a_{t_6}, x_{t_6})$

$$x_{e,t_6} = \dots + \sum f_3' x_{j,t_6}^{a_3} x_{k,t_6}^{-b_3}$$

by a very explicit rank 2 formula

(LS 2011) (same as GHKK rank broken line formula)

$f_3$  still can be factored

LS



LLZ

formula for rank 2  
greedy bases

GHKK



theta bases in  
rank 2

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