16/08/16

Kyungyong Lee: Positivity for cluster ~~algebras~~ variables

History:

Seiberg duality for quiver's (1994)

quiver = directed graph (multiple arrows poss.) no loops or 2-cycles

$$\begin{tikzcd} i \arrow[r] & j \arrow[r] & k \end{tikzcd} \xrightarrow[\text{at } j]{\text{duality}} \begin{tikzcd} i \arrow[rr, bend left] & j \arrow[l] & k \arrow[l] \end{tikzcd}$$

$$\begin{tikzcd} i \arrow[r] \arrow[rr, bend left] & j \arrow[r] & k \arrow[ll, bend right] \end{tikzcd}$$

cancel 2-cycles

certain moduli of quiver reps are preserved under duality.

(2000): Fomin and Zelevinsky discovered alg. structure for Seiberg duality.

$$\underbrace{\begin{matrix} i & \longrightarrow & j & \longrightarrow & k \\ f_i & & f_j & & f_k \end{matrix}}_{\text{rat. fct. on } x_1, x_2, \dots} \quad \xrightarrow[\text{at } j]{\mu_j \text{ mutation}} \quad \begin{tikzcd} i \arrow[rr, bend left] & j \arrow[l] & k \arrow[l] \end{tikzcd}$$

$$f_i \qquad \cancel{\frac{f_i + f_k}{f_j}} \qquad f_k$$

$$f_i(x_1', x_2', \dots) \qquad f_j(x_1', \dots) \qquad f_k(x_1', \dots)$$

where
$$x'_p = \begin{cases} x_p & \text{if } p \neq j \\ \dfrac{x_i + x_k}{x_j} & p = j \end{cases}$$

Let Q be a quiver with n vertices

$$\begin{array}{c} (Q, (x_{1,t_1}, \dots, x_{n,t_1})) \\ \mu_1 \swarrow \quad \dots \quad \searrow \mu_n \\ (Q_{t_2}, (x_{1,t_2}, \dots, x_{n,t_2})) \\ \mu_2 \swarrow \quad \dots \quad \searrow \mu_n \end{array}$$

Take an arbitrary seq. of mutations from $(Q, (x_{1,t_1}, \dots, x_{n,t_n}))$

$\Rightarrow$ the resulting seed $(Q', \underbrace{(f_1, \dots, f_n)}_{\text{"cluster variables"}})$

Example: $n=2$ $\quad Q = 1 \rightrightarrows 2$ (r arrows)

$$(Q, (x_{1,1}, x_{2,1})) \xrightarrow{\mu_1} \left(Q^{op}, \left(\frac{x_{2,2}^r + 1}{x_{1,2}}, x_{2,2}\right)\right)$$

$$\Big\downarrow \mu_2$$

$$\dots \xrightarrow{\mu_1} \left(Q, \left(\frac{\left(\frac{x_{1,3}^r + 1}{x_{2,3}}\right)^r + 1}{x_{1,3}}, \frac{x_{1,3}^r + 1}{x_{2,3}}\right)\right)$$

gets complicated very quickly

If $r=1$
$$\frac{\left(\frac{\frac{x_{24}+1}{x_{14}}+1}{x_{24}}\right)+1}{\frac{x_{24}+1}{x_{14}}} = \frac{(x_{24}+1)(x_{14}+1)}{(x_{24}+1)}\frac{x_{14}}{x_{14}x_{24}}$$
$$= \frac{x_{14}+1}{x_{24}}$$

factorization & cancellation occurs!

Theorem (FZ 2000)
Any cluster variable at any seed t is a Laurent polynomial of $x_{1t}, \dots, x_{nt}$ with integer coeff.

Conjecture (FZ 2000)
the coeff. are non-negative.

NOT trivial: $\frac{1+x^3}{1+x} = 1-x+x^2$

partial answers
acyclic: Nakajima (2009 ~ 2014)
Kimura - Qin (2012)
Surfaces: Musiker - Schiffler - Williams

Theorem (L-Schiffler 2013, Gross-Hacking-Keel-Kontsevich 2014)
The conjecture holds in full generality.

Both proofs
- have good control over rank 2 cluster variables
- use repeated applications (induction) of rank 2 computations

also, both formulas for rank 2 cluster variab. can be generalized to those for bases in rank 2 cluster algebras.

Taking a sequence of mutations. Locally looks like

$$\cdots \underbrace{\mu_j \mu_i \mu_j \mu_i \cdots \mu_i \mu_j}_{\text{rank } 2} \overbrace{\mu_k \mu_j \cdots \mu_k \mu_j}^{\text{rank } 2}$$

$$\begin{tikzcd}
(Q_{t_1}, (\cdots, x_{l,t_1}, \cdots)) \arrow[r, "\mu_l"] & \cdots \arrow[r, "\mu_j"] \arrow[d, dash, "\mu_k"] & (Q_{t_2}, X_{t_2}) \arrow[r, dash, "\mu_i \; \mu_j \cdots \mu_i \; \mu_j"] \arrow[d, dash, "\mu_k"] & (Q_{t_6}, X_{t_6}) \arrow[d, dash, "\mu_k"] \\
 & (Q_{t_5}, X_{t_5}) & (Q_{t_3}, X_{t_3}) \arrow[d, dash, "\mu_j"] & \cdot \arrow[d, dash, "\mu_j"] \\
 & & (Q_{t_4}, X_{t_4}) & \vdots
\end{tikzcd}$$

$$x_{l,t_1} = \sum_{a_1, b_1 \in \mathbb{N}} f_1 \, x_{j,t_2}^{-a_1} x_{k,t_2}^{-b_1} + \cdots + \sum_{a_3, b_3 \in \mathbb{N}} f_3 \, x_{j,t_2}^{a_3} x_{k,t_2}^{b_3}$$

where f_i are Laurent poly of

$x_{1,t_2}, \ldots, \hat{x}_{j,t_2}, \ldots, \hat{x}_{k,t_2}, \ldots, x_{n,t_2}$

Induction hypothesis: (on the # of rank 2 subseq.)

f_3 can be factored into

$$\underbrace{g_3}_{\text{pos. Laurent}} \Big(\prod_{a \to k} x_{a,t_2} + \prod_{b \leftarrow k} x_{b,t_2} \Big)^{b_3}$$

$$\Rightarrow \quad f_3 \, x_{j t_2}^{a_3} \, x_{k t_2} = g_3 \, \underbrace{x_{j t_3}^{a_3} \, x_{k t_3}^{b_3}}_{\text{pos. along } \mu_j \mu_k}$$

After $\mu_i \mu_j \cdots \mu_j \, (Q_{t_6}, X_{t_6})$

$$x_{e,t_1} = \cdots + \sum f_3' \, x_{j t_6}^{a_3} \, x_{k t_6}^{-b_3}$$

by a very explicit rank 2 formula (LS 2011) (same as GHKK rank broken line formula)

f_3 still can be factored

$$\begin{array}{ccc}
\text{LS} & & \text{GHKK} \\
\downarrow & & \downarrow \\
\text{LLZ} & & \\
\text{formula for rank 2 greedy bases} & = & \text{theta bases in rank 2}
\end{array}$$