10/08/16 ICRA

Robert Marsh

Dimer models & Cluster categories of Grassmannian

1. Cluster algebra structure of Gr.
   J. Scott
   Gekhtman - Shapiro - Vainshtein
2. Jensen - King - Su categorification
3. Baur - King - Marsh dimer models describe endom. alg of cluster tilting object

1. Grassmannian

Fix $k \leq n \in \mathbb{N}$

$X = Gr(k,n)$ Grassmannian of k-dim subspaces of $\mathbb{C}^n$

$X \ni$ a $k \times n$ matrix of rank k,
rows = spanning vectors

Two matrices a, a' corresp. to same subspace $\Leftrightarrow \exists\, g \in GL_k\mathbb{C}$ with $ga = a'$

If $I \subseteq [n] = \{1, \dots, n\}$, $|I| = k$

then $\Delta_I(a)$ = minor of a with rows $1, \dots, k$ & colums I

Plücker coordinate

Replacing a with ga scales all Plückers by same scalar

Get well-def. map

$$X \hookrightarrow \mathbb{P}^{\binom{n}{k}-1} \quad \text{proj. space}$$

$$a \longmapsto (\Delta_I(a))_I \quad \text{Plücker embedding}$$

The image is proj. var. def. by Plücker relations. This includes Plücker relations

$$\Delta_{Jac}\,\Delta_{Jbd} = \Delta_{Jab}\,\Delta_{cd} + \Delta_{Jad}\,\Delta_{Jbc}$$

where $J \subset [n]$ of size $k-2$ disjoint from a, b, c, d. $Jab = J \cup \{a, b\}$ etc.

a, b, c, d cyclic in $[n]$

By transfer of structure X is proj var.

$\hat{X}$ = aff cone of X

aff. subvar of $\mathbb{C}^{\binom{n}{k}}$ def. by same polys

The homogeneous coord ring of X is
coord ring $\mathbb{C}[\hat{X}]$ of $\hat{X}$.

$$\mathbb{C}[X] \underset{\text{def.}}{=} \mathbb{C}[\hat{X}] = \underbrace{\mathbb{C}[\Delta_I \mid I \subset [n], |I| = k]}_{\text{Plücker relations}}$$

Function field of $\hat{X}$ = field of functions of
$\underbrace{\mathbb{C}[X]}$ integral domain

= field of rational fct over $\mathbb{C}$ in $k(n-k)+1$
indeterminants

2. Cluster Algebras

[Fomin Zelevinsky 2002]

Fix natural numbers $N, M \in \mathbb{N}$

$\mathbb{F} = \mathbb{C}(u_1, \dots, u_{N+M})$

$\underline{C} \subseteq \mathbb{F}$, $|\underline{C}| = M$ alg. indep. (frozen var.)

A <u>seed</u> in $\mathbb{F}$ is a pair $S = (\underline{x} \cup \underline{C}, Q)$

- $\underline{x} \cup \underline{C}$ free gen. set for $\mathbb{F}$ over $\mathbb{C}$
  <u>extended cluster</u>
  $\underline{x} \cap \underline{C} = \emptyset$
  $\underline{x}$ <u>cluster</u>

- Q cluster quiver with vertices $\underline{x} \cup \underline{C}$

quiver with no loops or 2-cycles

Given $x \in \underline{X}$ can mutate S at x to get new seed

$$\mu_x S = \left((\underline{x} \setminus \{x\}) \cup \{x'\} \cup \underline{C}, \ \mu_x Q \right)$$

where
$$x x' = \prod_{\substack{y \to x \\ \text{in } Q}} y + \prod_{\substack{x \to y \\ \text{in } Q}} y$$

with multiplicity

and $\mu_x Q$ is obtained from Q by the following

(a) $\forall$ $a \to x \to b$ add $a \to b$ (with mult.)

(b) Remove maximal set of two cycles

(c) Reverse all arrows incident with x

– quiver mutation

$\mathbb{S}$ = set of seeds obtained from S by arbitrary fin. seq. of mutations

initial seed

$\mathcal{X}$ = union of clusters in seeds in $\mathbb{S}$

"cluster variables"

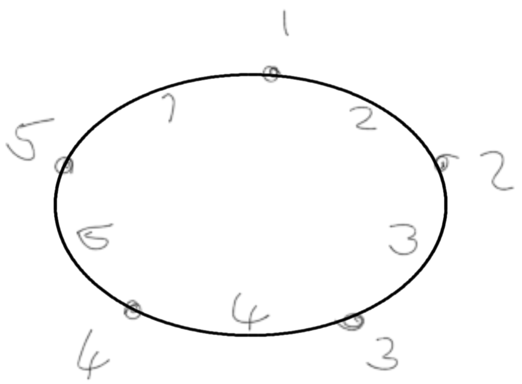
$\mathcal{A}(S)$ = $\mathbb{C}$-subalg. of $\mathbb{F}$ gen. by $X \cup C$
"cluster algebra"

Up to (strong) isom. of cluster algebras
$\mathcal{A}(S)$ depends only on Q
- denote it $\mathcal{A}(Q)$

3. Postnikov diagrams

C circular graph
vertices $C_0 = \mathbb{Z}_n$ clockwise
edges $C_1 = \mathbb{Z}_n$ (another copy)

$$i-1 \overset{i}{\text{———}} i$$



A (k,n)-Postnikov diagram D consists of n directed curves (strands) in disk with n marked vertices on its boundary.

Strand $i \rightsquigarrow i+k$

local axioms:

(a1) Only 2 strands cross in given pt
All crossings transversal

(a2) fin. many crossings pts

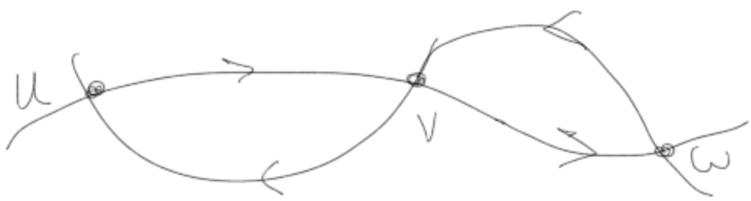
(a3) Proceeding along a strand. the other strands crossing it alternate btw left to right and right to left.

start & end are also regarded as crossing.

global axioms:

(b1) no self intersections

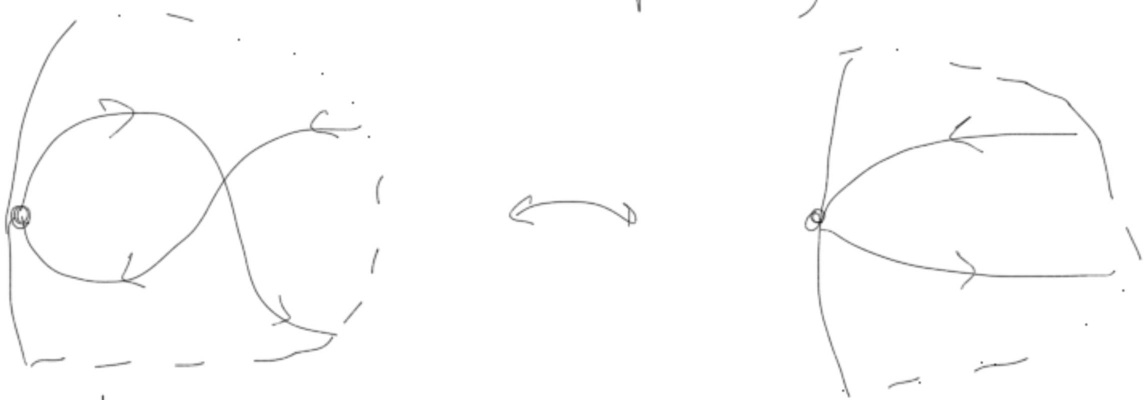
(b2) If two strands cross at u, v, $u \neq v$ then one strand is oriented $u \to v$ & the other $v \to u$

eg. 

↳ lens

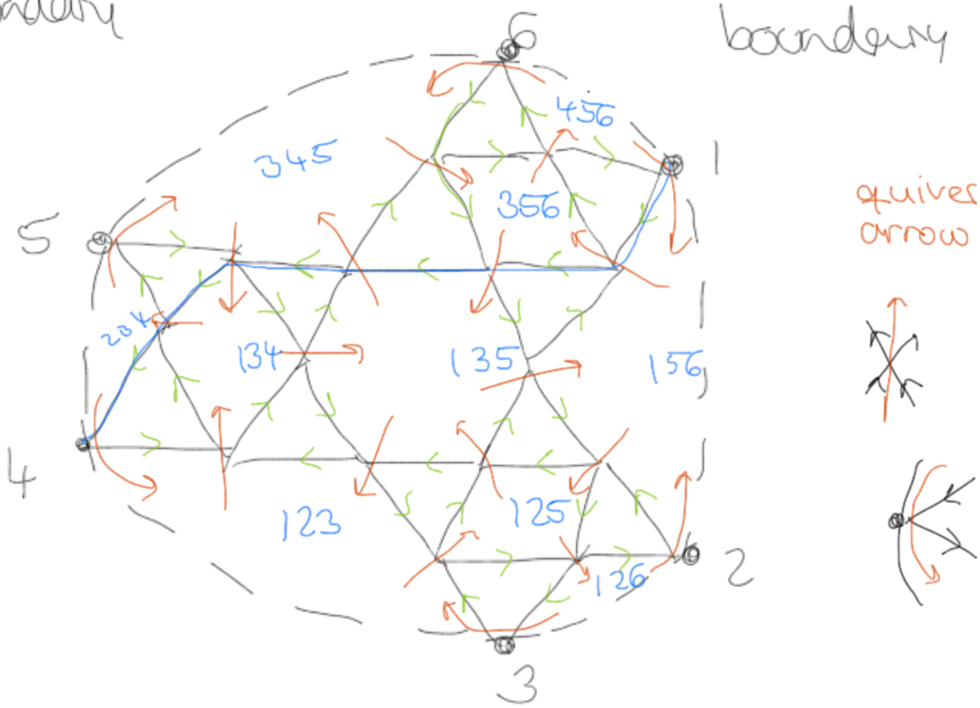
Consider these up to isotopy. and twisting / untwisting moves.

local move (no other strand in this picture)



boundary   boundary

Exp.



get regions of three kinds

oriented clockwise } triangles in Exp.
— " — anticlockwise

alternating – rhombi, hexagon boundary regions label alternating regions wil strands passing to right

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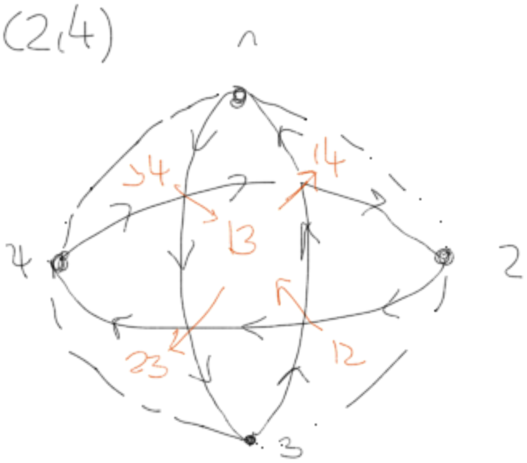
We fixed $k, n \in \mathbb{N}$, $k \leq n$, $Gr(k,n)$

Defined (k,n)-Postnikov diagram D

<u>Eg</u>: $Gr(2,4)$



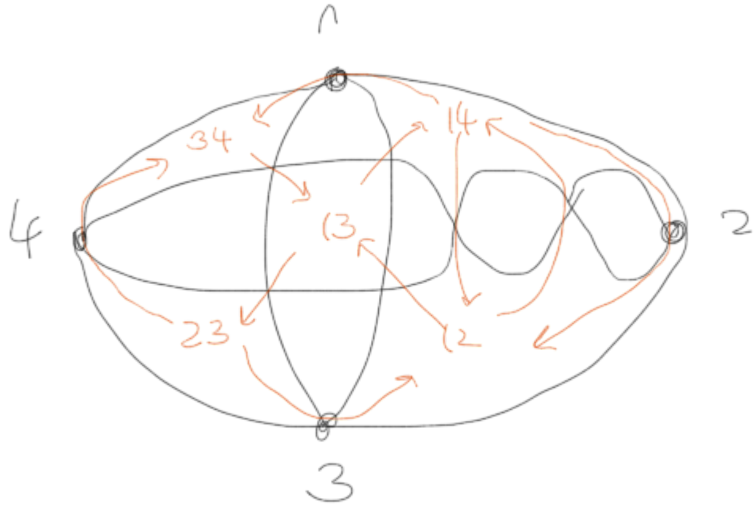
$C = \{ \Delta_I \mid I \text{ labels boundary alternating region} \}$

$= \{ \Delta_{E_i} \mid i \in [1,n] \}$ $\quad E_i = [i-k+1, i]$ cyclic

$X_D = \{ \Delta_I \mid I \text{ labels internal alternating region} \}$

Q_D quiver with vertices $C \cup X_D$

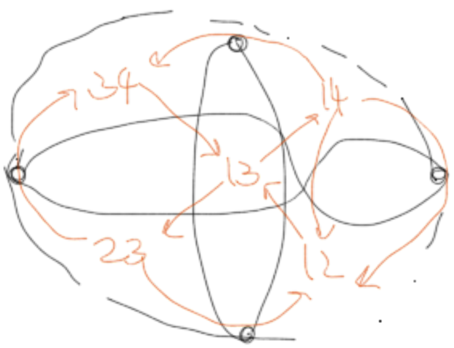
and arrows



could have 2-cycles
$\rightsquigarrow$
reduce

$\rightsquigarrow$ seed $S_D = (\underline{c} \cup \underline{x}_D, Q_D)$ in $\mathbb{C}(X)$

$\tilde{Q}_D$ = quiver with boundary arrows



can reduce boundary twist

<u>Theorem</u> (Scott: Gr & CA's, 2006)

$\mathbb{C}[X]$ is a cluster algebra

Initial seed S_D for some (k,n)-Postnikov diag D.

The $S_{D'}$ for all D' are seeds.

All Plücker coordinates are cluster variables.

see Gekhtman-Shapiro-Vainshtein also

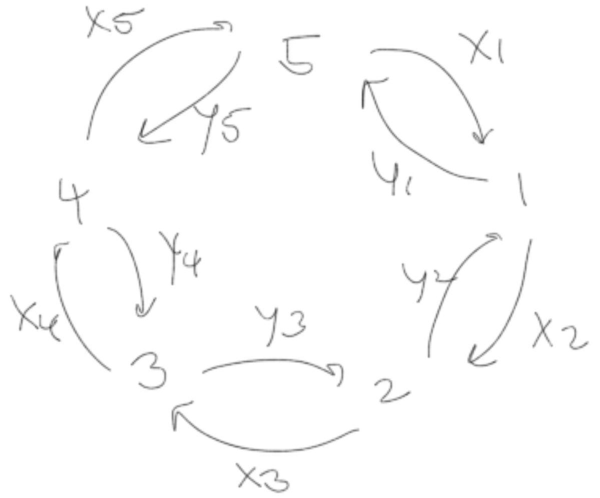
4. JKS categorification

First categorification done by GLS (almost complete)

$Q(C)$ quiver $\longleftrightarrow$ C

$Q_0 = C_0$ vertices $\qquad x_a : a-1 \to a$

$Q_1 = \{ x_a, y_a \mid a \in C_1 \}$ $\qquad y_a : a \to a-1$



$$\overline{B} = \mathbb{C}Q \Big/ \begin{pmatrix} xy = yx \\ y^{n-k} = x^k \end{pmatrix} \longrightarrow \text{for every } i \quad x_{i+1} y_{i+1} = x_i y_i$$

$\hat{B}$ = completion of $\overline{B}$ wrt arrow ideal

Rmk: Krull-Schmidt holds for $\hat{B}$-mod
not known for $\overline{B}$.

$\bar{B}, \hat{B}$ are Gorenstein (Buchweitz)

↳ left & right Noeth. & fin. left & right inj. dim

If $B = \bar{B}$ or $\hat{B}$ can take $CM(B)$ max. Cohen Macaulay modules $= \{ M \mid \mathrm{Ext}^i(M, B) = 0 \ \forall i > 0 \}$

↳ "projectives & injectives coincide"

$CM(B)$ is Frobenius category with

proj. inj.'s = proj. B-modules

Frobenius cat. = exact category (have SES) with enough proj. & inj., proj. = inj.

$\mathcal{F} = CM(\hat{B})$ Extended clusters in $\mathbb{C}[X]$ are modelled by cto's in $\mathcal{F}$.

(same def. as in Claire's talk)

proj. inj. are summands of every cto.

If $T = U \oplus X$ in $\mathcal{F}$ is cto with X indec. & <u>not</u> proj.

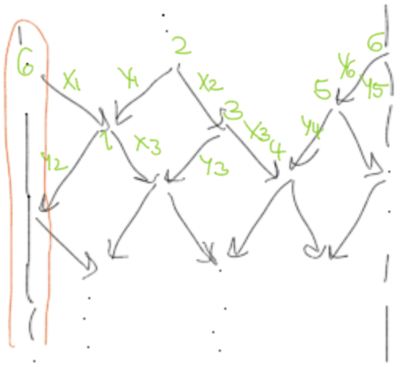
Then $\exists$ unique indec. X' in $\mathcal{F}$ s.t. $T' = U \oplus X'$ is cto

T' is <u>mutation</u> of T at X.

Plücker coord. $\Delta_I \rightsquigarrow M_I$ in $\mathcal{F}$

Ex. $I = \{1,3,4\}$ $k=3, n=6$

$\hat{M}_I$ in $CM(\hat{B})$

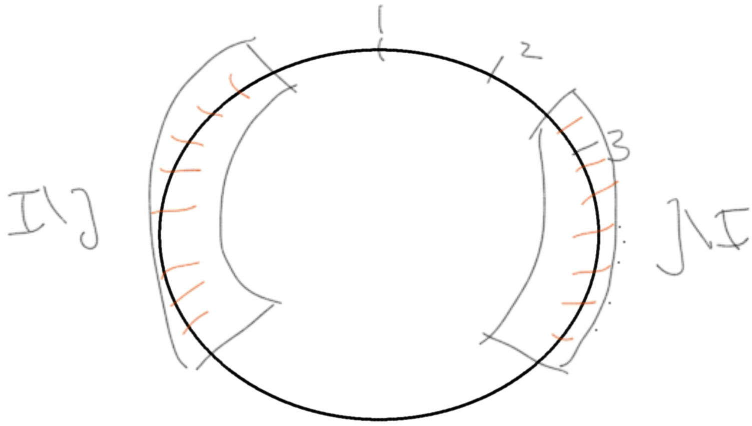


$$k \xrightarrow{x} k \quad \text{id}$$

Column corresp. to vector space at vertex 6

↳ take columns 1,3,4 for I

Two k-subsets I, J of [n] are <u>non-crossing</u> (or weakly separated) if ∄ cyclically ordered a,b,c,d with $a,c \in I \setminus J$ and $b,d \in J \setminus I$.



<u>Theorem</u> (JKS)

(a) ∀ k-subsets I, J of [1,n]

$$\operatorname{Ext}(M_I, M_J) = 0 \iff I, J \text{ non crossing}$$

(b) [Postnikov, Scott, Oh-Postnikov-Speyer]

The maximal collections of noncrossing subsets of [n] are precisely those arising from (k,n)-Postnikov diag. In particular, all have cardinality $k(n-k)-1$

(c) [JKS] D (k,n) Postnikov diag.

$$T_D = \sum_{I \text{ label}} M_I \;, \quad \hat{T}_D = \sum_{I \text{ label}} \hat{M}_I$$

$\hat{T}_D$ is a cto in $\tilde{\mathcal{F}}$

A cto is <u>reachable</u> if it can be reached from $\hat{T}_D$ (equiv. all $\hat{T}_D$'s) by a finite sequence of mutations.

A rigid indec. in $\mathcal{F}$ is reachable if it is a direct summand of those.

<u>Theorem</u> (JKS)

(1) $\exists$ map $ob(\tilde{\mathcal{F}}) \longrightarrow \mathbb{C}[X]$ satisfying

$$M \longmapsto \psi_M$$

a) ψ_M depends only on M (upto isom.)

b) $\psi_{M_1 \oplus M_2} = \psi_{M_1} \psi_{M_2}$

c) If $\dim \operatorname{Ext}_{\mathcal{F}}(X,Y) = 1 = \operatorname{Ext}_{\mathcal{F}}(Y,X)$ and

$$Y \hookrightarrow E \twoheadrightarrow X$$
$$X \hookrightarrow F \twoheadrightarrow Y$$

are the corresp. non split seq. then

$$\psi_X \psi_Y = \psi_E + \psi_F$$

a), b), c) : ψ is a cluster character (Caldero Chapoton)

(<u>Rmk</u>: $\mathcal{F}$ is stably 2-CY, i.e. stable cat $\underline{\mathcal{F}}$ is 2-CY)

(2) $M \mapsto \varphi_M$ induces bijections

$$\left\{ \begin{array}{c} \text{isoclasses of} \\ \text{reachable rigid} \\ \text{indec. in } \mathcal{F} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{cluster variables} \\ \text{of } \mathbb{C}[X] \end{array} \right\}$$

$$\hat{M}_I \longleftrightarrow \Delta_I$$

$$\left\{ \begin{array}{c} \text{isoclasses of} \\ \text{reachable cto's} \\ \text{in } \mathcal{F} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{extended clusters} \\ \text{in } \mathbb{C}[X] \end{array} \right\}$$

$$\hat{T}_D \longleftrightarrow \begin{array}{c} \text{extended cluster from } D \\ \underline{c} \cup \underline{x}_D \end{array}$$

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$$\left\{\begin{array}{l}\text{isom classes} \\ \text{of reachable} \\ \text{cluster tilting obj in } \mathcal{F} = CM(\hat{B})\end{array}\right\}$$

$$\updownarrow \text{ JKS}$$

$$\left\{ \text{(extended) clusters of } \mathbb{C}[X] \right\}$$

$$\updownarrow \text{ Gekhtman Shapiro Vainshtein 2008}$$

$$\left\{ \text{seeds of } \mathbb{C}[X] \right\}$$

5. 2-CY-tilted algebras

- what are the algebras $\mathrm{End}_{\mathcal{F}}(\hat{T}_D)$ for D Postnikov diagram?
- quiver with relations

A <u>dimer model</u> is a quiver embedded in a surface with complement = union of disks

Boundary of each disk = cycle in the quiver

For a surface with boundary, each boundary component should be a cycle in the underlying quiver.

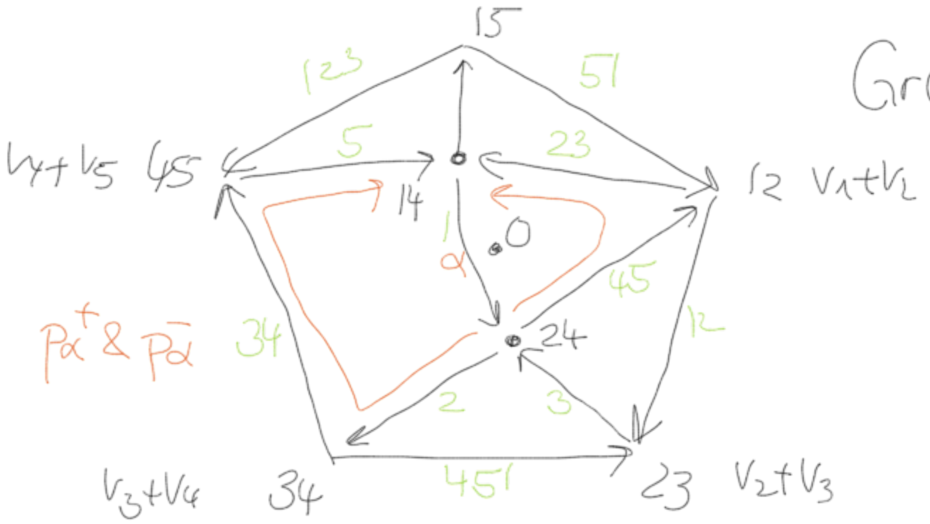
So $\bar{Q}_D$ is an example of a dimer model with boundary in the disk.

$\tilde{Q}_2 = \{\text{discs in complement of } \tilde{Q}_0\}$

$F \in \tilde{Q}_2 \rightsquigarrow \partial F \in \tilde{Q}_{cyc} = \{\text{(oriented) cycles of } \tilde{Q}_0\}$

$F^+ = \{F \in \tilde{Q}_2 \mid \partial F \text{ anticlockwise}\}$

$F^- = \{F \in \tilde{Q}_2 \mid \partial F \text{ clockwise}\}$

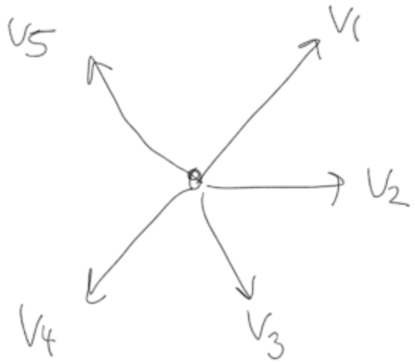


Gr(2,5)

Labels see later →

Oh-Postnikov-Speyer & [BKM]

plot



vectors

Q dimer model with boundary $\rightsquigarrow$ dimer algebra A_Q

$\alpha \in Q_1$ internal arrow $\exists$ unique faces $F^+ \in Q_2^+$ and $F^- \in Q_2^-$

s.t. $\partial F^+ = \alpha p_\alpha^+$ $\quad p_\alpha^+ : \text{head}(\alpha) \longrightarrow \text{tail}(\alpha)$

$\partial F^- = \alpha p_\alpha^-$ $\quad p_\alpha^- :$

$A_Q = \mathbb{C}Q / (p_\alpha^+ = p_\alpha^-)$

Remark: Set $W_Q = \sum_{F \in Q_2^+} \partial F - \sum_{F \in Q_2^-} \partial F$

then $A_Q = \mathbb{C}Q / (\partial_\alpha W_Q \mid \alpha \text{ internal})$

↳ kind of frozen Jacobi algebra

set of frozen arrows → don't diff. them (no relation)

Theorem [BKM]

D (k,n)-Postnikov diagram

1 a) $\mathrm{End}_{\overline{B}}(T_D) \xleftarrow{f}{\cong} A_D = A_{\tilde{Q}_D}$

b) $\mathrm{End}_{\hat{B}}(\hat{T}_D) \cong \hat{A}_D$

2 Let $e = \sum_{i=1}^{n} e_{E_i}$

Boundary algebra $eA_De \cong \overline{B}^{op}$

$e\hat{A}_De \cong \hat{B}^{op}$

Demonet - Luo

$k=2$ Disc & punctured disc

Gorenstein orders

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1 b) follows from a) $u = \sum_{I \in D}$ cycle at I

$\in$ center of A_D

$\hat{A}_D = A_D \otimes_{\mathbb{C}[u]} \mathbb{C}[[u]]$

$t = \sum x_i y_i \in$ center of $\overline{B}$

$\hat{B} = B \otimes_{\mathbb{C}[t]} \mathbb{C}[[t]]$

$\hat{M}_I = M_I \otimes_{\mathbb{C}[t]} \mathbb{C}[[t]]$

(2) $e A_D e \underset{1(a)}{\cong} f(e) \operatorname{End}_{\overline{B}}(T_D) f(e)$

$\cong \operatorname{End}_{\overline{B}}(P)$

where $P = \bigoplus_{i=1}^{n} M_{E_i} \cong {}_B\overline{B} \cong \overline{B}^{op}$

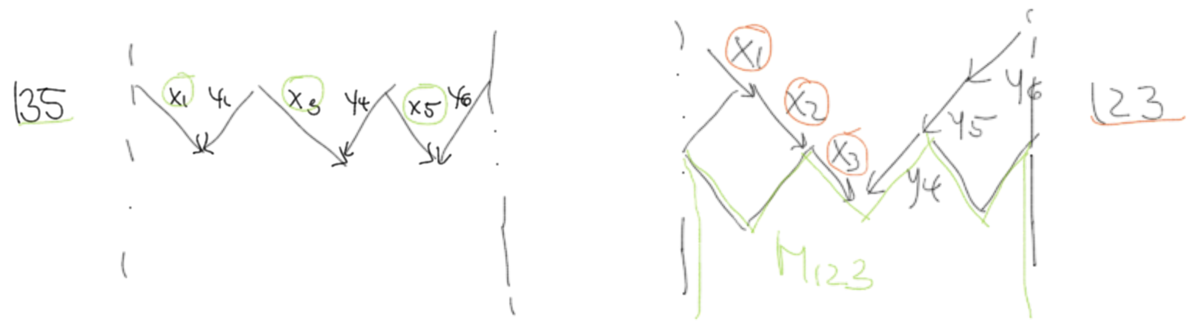
↳ left to proof (1) a)

I, J k-subsets of $[1,n]$

$\operatorname{Hom}_{\overline{B}}(M_I, M_J)$. Have maximal embedding

$\varphi_{JI} : M_I \longrightarrow M_J$

e.g. $I = 135$, $J = 123$



$f : A_D \longrightarrow \operatorname{End}_A(T_D)$

$e_I \longmapsto id_{M_I}$

$(I \xrightarrow{\alpha} J) \longmapsto (\varphi_{JI} : M_I \longrightarrow M_J)$

$\mathrm{Hom}_B(M_I, M_J) \cong \mathbb{C}[t]$ $\quad t \longleftrightarrow$ "shift down by 1"

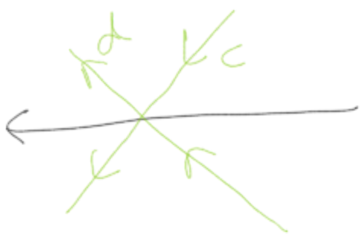
$\varphi_{JI} \longleftrightarrow 1 \in \mathbb{C}[t]$

Morphisms $\longleftrightarrow$ powers of t "homogeneous"

φ is called <u>sincere</u> if φ homogeneous and coker φ is sincere (supp. on all vertices)

φ_{JI} is the unique insincere hom. $M_I \longrightarrow M_J$

<u>Check relations</u>: Label arrows of $\tilde{Q}_D$



Label $= [c, d-1] \subseteq [1, n]$
w_α cyclic interval
regarded as element of $\mathbb{N}C_0$

$$[c, d-1] = (c,d)_0 = \text{vertices incident with } (c,d) \subseteq C_1$$

<u>Lemma</u>

1. <u>dim</u> coker $\varphi_{IJ} = w_\alpha$
2. [OPS, BKM] $F \in \tilde{Q}_2(D)$ face

$$\sum_{\alpha \in \partial F} w_\alpha = [1, n] = \sum_{i=1}^{n} e_i \in \mathbb{N}C_0$$

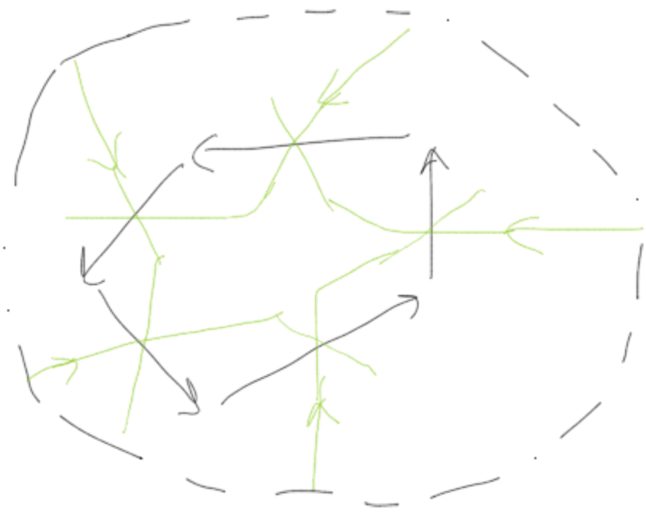
$$\begin{tikzcd} J_2 \arrow[d] & I_1 \arrow[l] \arrow[r] & I_2 \arrow[d] \\ \vdots \arrow[d] & & \vdots \arrow[d] \\ J_s \arrow[r] & I_0 \arrow[u] & I_r \arrow[l] \end{tikzcd}$$

$\varphi_{I_0 I_r} \circ \varphi_{I_r I_{r-1}} \circ \cdots \circ \varphi_{I_2 I_1}$ is insincere homog. morph $I_1 \to I_0$

$= \varphi_{I_0 I_1}$

Consistency condition:

not allowed



strands hit boundary in same order

$\rightsquigarrow$ have well-def map: check inj & surj

<u>Surjectivity</u>: $f : A_D \longrightarrow \mathrm{End}_{\overline{B}}(T_D)$

$? \longmapsto \varphi_{JI}$

need minimal path

e.g. (from before)

$15 \to 45 \to 14 \to 24 \to 12$ sincere but no cycle

only "hidden" cycle

not visable directly

Given I, J vertices in D construct minimal path (insincere) p_{JI} in $\tilde{Q}_D$ I to J

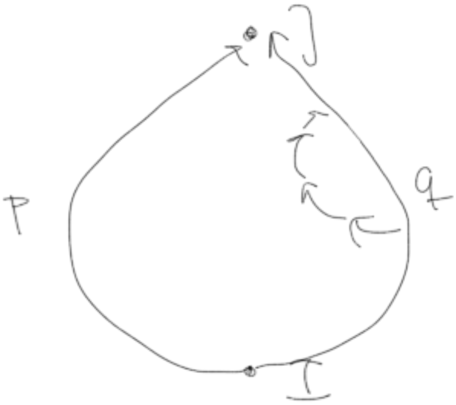
so $f(p_{JI}) = \varphi_{JI}$ will be a homog. insincere morph.

To construct it, need consistency condition and

$$\sum_{\substack{\alpha \text{ incident} \\ \text{with } I}} w_\alpha = (r-1)[1,n]$$

internal with $2r$ arrows incident with I

<u>Injectivity</u>: $e_J A_D e_I$



want to apply rel.

(if p, q both min $p = q$ for example)

show $\exists!$ insincere path $I \to J$ then every path equal to $u^r \cdot p_{JI}$ for $r \geq 0$, u = cycle at J

Bocklandt

<u>Rmk</u>: Fomin-Pylavskyy configuration of pts in V & (V^*)

$k\,(\,\hat{|}\;|\;\cdots\;|\,)$