11/08/16

Recall $\mathcal{T} = \mathrm{StMod}\,G$, $\mathcal{T}^c = \mathrm{stmod}\,G$

<u>Def</u>: $\mathcal{C} \subset \mathrm{stmod}\,G$ is <u>thick</u> if
1) $\mathcal{C}$ is full triang. subcat
2) $\mathcal{C}$ is closed under $\oplus$ summands

<u>Def</u>: $\mathcal{C} \subset \mathrm{StMod}\,G$ is <u>localizing</u> if
1) $\mathcal{C}$ is full triang. subcat
2) $\mathcal{C}$ is closed under $\oplus$ summands

<u>Def</u>: $\mathcal{C} \subset \mathrm{StMod}\,G$ is <u>$\otimes$-ideal</u> if
$\forall\ M \in \mathcal{C},\ N \in \mathrm{StMod}\,G \quad M \otimes_k N \in \mathcal{C}$

<u>Lecture 2: Support theories for St Mod G</u>

<u>II. 1. Informal problem</u>

G fin. grp scheme, k field, char $k = p$

rep G wild (except $\mathbb{Z}/p\mathbb{Z}$, $\alpha_p = \mathbb{G}_{a(1)}$ fin.)

eg $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ already wild
& $\mathbb{G}_{a(2)}$

↳ cannot classify up to direct sums

↳ classify up to homolog. operations

- extensions
- syzygies
- dir. sums / dir. summands

} $*$

$M, N \in \mathrm{Rep}\, G$

Q: When can we build M out of n using $*$

↳ still too hard

↳ also allow tensoring with simple obj.

Answer: can build M out of N

$\iff$ $\boxed{\mathrm{supp}\, M \subset \mathrm{supp}\, N}$

Plan: define $\mathrm{supp}\, M$ and describe properties.

proof in Lecture 3.

II.2. the formulation of the problem

StMod G — objects as in $\mathrm{Rep}\, G$

morph $\dfrac{\mathrm{Hom}_G(M,N)}{\mathrm{PHom}_G(M,N)}$

those which factor through proj. obj.

stmod G – fin dim. modules only

$\text{stmod } G \subseteq (\text{StMod } G)^c$

$$D^{perf}(\text{rep } G) \longrightarrow D^b(\text{rep } G) \longrightarrow D_{sing}(G)$$

$D_{sing}(G) \cong \text{stmod } G$ Rickard, Buchweitz

StMod G is $\otimes$ (tensor triangulated)

shift Ω^{-1}

$$M \hookrightarrow I(M) \twoheadrightarrow \Omega^{-1}(M)$$

Problem: Classify thick tensor ideals in stmod G.
Classify localizing tensor ideals in StMod G

need ∞-dim modules here.

II.3 History

1. Stable homotopy theory
   Devinatz – Hopkins – Smith
2. A commut. ring
   $K^b(\text{proj} A)$, $D(A)$ Hopkins, Neeman
3. Alg. geom.
   $D^{perf}(X)$ Thomason
4. G finite group
   stmod G Benson–Carlson–Rickard 97

StMod G 2010 Benson-Iyengar-Krause (BIK)

[different methods]

We'll need two support theories:

Need to work for ∞-dim modules

- BIK local cohomology support [classical hom. supp. var.]

- Π-support [Carlson's rank varieties for elem. ab. grp's]

required properties:

1) "2 out of 3" $M_1 \to M_2 \to M_3 \to$

$\Rightarrow$ $\operatorname{supp} M_2 \subset \operatorname{supp} M_1 \cup \operatorname{supp} M_3$

2) $\operatorname{supp}(M \oplus N) = \operatorname{supp} M \cup \operatorname{supp} N$

3) Shift: $\operatorname{supp} M = \operatorname{supp} \Omega^{-1} M$

4) Detection: $\operatorname{Supp} M = \emptyset \Leftrightarrow M \cong 0$ (M is proj)

5) $\operatorname{supp}(M \otimes N) = \operatorname{supp} M \cap \operatorname{supp} N$

6) Realization: $\forall\ Z \subset X$ closed $\exists$ fin. dim. G-mod M s.t. $\operatorname{supp} M = Z$ [Tensor]

where $X = \operatorname{Proj} H^*(G, k)$

II.4. BIK local cohom functors

cohom. support variety

$M \in \operatorname{rep} G$

$$H^*(G,k) \times \operatorname{Ext}^*_G(M,M) \longrightarrow \operatorname{Ext}^*(M,M)$$

$$I_M = \operatorname{Ann}_{H^*(G,k)}\left(\operatorname{Ext}^*_G(M,M)\right)$$

cohom. supp. $M = V(I_M) \subset \operatorname{Proj} H^*(G,k)$

$$I_M = \ker\left(H^*(G,k) \xrightarrow{-\otimes M} \operatorname{Ext}^*_G(M,M)\right)$$

Alperin – Evens, Carlson

<u>BIK theory</u>:

ring R acts on T $\otimes$

here $R = H^*(G,k)$ and $T = \operatorname{StMod} G$, $X = \operatorname{Proj} R$

$\mathfrak{p} \in X \rightsquigarrow \Gamma_{\mathfrak{p}} k$ universal local cohom. module

$$\underline{\operatorname{Hom}}^*(M,N) = \bigoplus_{i \in \mathbb{Z}} \underline{\operatorname{Hom}}(M, \Omega^{-i} N) \quad \left(\cong \widehat{\operatorname{Ext}}^*_G(M,N)\right)$$

$R = H^*(G,k)$ acts on $\underline{\operatorname{Hom}}^*(M,N)$

<u>Terminology</u>: Y R-module then Y is <u>$\mathfrak{p}$-local</u> ($\mathfrak{p} \in X$)

if $Y \longrightarrow Y_{\mathfrak{p}} = Y \otimes_R R_{\mathfrak{p}}$ is invertible

Y is <u>$\mathfrak{p}$-torsion</u> if $Y_{\mathfrak{q}} = 0$ $\forall\, \mathfrak{q} \notin V(\mathfrak{p})$ $(\mathfrak{p} \not\subset \mathfrak{q})$

In StMod G, $M \in$ StMod G

Def: M is p-local ($p \in X$) if $\underline{\mathrm{Hom}}^*(C,M)$ is p-local for any $C \in$ stmod G

$$\mathrm{StMod}\,G \underset{\longleftarrow}{\xrightarrow{\exists \text{ left adj}}} (\mathrm{StMod}\,G)_p = \left\{ \begin{array}{c} \text{all p-local} \\ \text{objects} \end{array} \right\}$$

$$M \longrightarrow M_p$$

⊗-ideal localizing subcat.

$$\underline{\mathrm{Hom}}^*(C,M)_p \xrightarrow{\sim} \underline{\mathrm{Hom}}^*(C, M_p)$$

Motivation: $\mathrm{supp}\, M_p \subset \{ q \in X \mid q \subset p \}$

Def: M is p-torsion if

$\underline{\mathrm{Hom}}^*(C,M)_q = 0$

$\forall\, q \notin V(p) \quad \forall\, C \in$ stmod G

$$\underbrace{\Gamma_{V(p)}\,\mathrm{StMod}\,G}_{\text{p-torsion objects}} \underset{\longrightarrow}{\xleftarrow{\exists \text{ right adj}}} \mathrm{StMod}\,G$$

$$\Gamma_{V(p)} M \longleftarrow M$$

Motivation: $\mathrm{supp}\, \Gamma_{V(p)} M \subset V(p)$

Def: $\Gamma_p M = \Gamma_{V(p)}(M_p) = (\Gamma_{V(p)} M)_p$

$M \in$ StMod G

$$\Gamma_p : \mathrm{StMod}\,G \longrightarrow \mathrm{StMod}\,G$$

$$\Gamma_p(\mathrm{StMod}\,G) = \{\text{p-local \& p-torsion objects}\}$$

$\Gamma_p M = \Gamma_p k \otimes M$

$\Gamma_p k$ local cohom. module

$\Gamma_p k = \mathcal{K}$-modules (Richard idempotent modules)

Def: $M \in \mathrm{StMod}\, G$

$$\mathrm{supp}_{BIK}(M) = \{ p \in X \mid M \otimes \Gamma_p k \not\simeq 0 \}$$

$$\mathrm{cosupp}_{BIK}(M) = \{ p \in X \mid \mathrm{Hom}_k(\Gamma_p k, M) \not\simeq 0 \}$$

Remark $M \in \mathrm{stmod}\, G \Rightarrow$ cohom $\mathrm{supp}\, M = \mathrm{supp}_{BIK} M$
$= \mathrm{cosupp}_{BIK} M$

All different for $M \in \mathrm{StMod}\, G$

[Table *]

	$\mathrm{supp}_{BIK} M$
1	✓
2	✓
3	✓
4	✓
5	Problem!
6	✓

II.5. Rank varieties

$E = (\mathbb{Z}/p)^n \qquad kE = \dfrac{k[x_1, \dots, x_n]}{(x_1^p, \dots, x_n^p)} \qquad p \neq 2$

$H^*(E, k) = k[y_1, \dots, y_n] \otimes \Lambda(t_1, \dots, t_n)$

$X = \mathbb{P}^{n-1} \qquad k[y_1, \dots, y_n]$ homog. coord. alg. of $\mathbb{P}^{n-1}$

$\underline{\lambda} = [\lambda_1 : \dots : \lambda_n] \in \mathbb{P}^{n-1}$

$x_{\underline{\lambda}} = \lambda_1 x_1 + \dots + \lambda_n x_n \in kE$

$\langle x_{\underline{\lambda}} + 1 \rangle \cong \mathbb{Z}/p\mathbb{Z}$ "cyclic shift subgrp of E"

$M \downarrow_{\langle x_{\underline{\lambda}} + 1 \rangle}$

Theorem (Avrunin–Scott '82)

$M \in \operatorname{rep} G$

$$\underset{\substack{\cap \\ X = \mathbb{P}^{n-1}}}{\text{cohom supp } M} \cong \left\{ \underline{\lambda} \in \mathbb{P}^{n-1} \,\middle|\, M\downarrow_{\langle x_{\underline{\lambda}}+1 \rangle} \text{ not free} \right\}$$

Lecture 3: π-support, π-cosupport and classification for StMod

jt BIK = Benson, Iyengar, Krause

$X = \operatorname{Proj} H^*(G,k)$

Recall:

$E = (\mathbb{Z}/p)^n \qquad kE = k[x_1, \ldots, x_n]/(x_1^p, \ldots, x_n^p)$

$\underline{\lambda} \in \mathbb{P}^{n-1} \longmapsto x_{\underline{\lambda}} = \lambda_1 x_1 + \cdots + \lambda_n x_n \in kE$

$\langle x_{\underline{\lambda}} + 1 \rangle \hookrightarrow kE$ cyclic shifted subgroup

$\cong$

$\mathbb{Z}/p$

Theorem: cohom supp $M = \{ \underline{\lambda} \in \mathbb{P}^{n-1} \mid M\downarrow_{\langle x_{\underline{\lambda}}+1 \rangle} \text{ not free} \}$

$\mathfrak{g}$ restricted Lie algebra

$N_p(\mathfrak{g}) = \langle x \in \mathfrak{g} \mid x^{[p]} = 0 \rangle$

$\cup$

$\mathfrak{g} = \operatorname{Spec} S^*(\mathfrak{g}^\#)$

$\operatorname{Proj} H^*(\mathfrak{g}, k) \simeq \operatorname{Proj} N_p(\mathfrak{g})$

<u>Theorem</u>: (Suslin-Friedlander-Bendel, 1997)

$$X \supset \text{cohom supp}\, M = \operatorname{Proj}\{x \in \mathcal{N}_p(\mathfrak{g}) \mid M|_{\langle x \rangle} \text{ not free}\}$$

<u>Prototype</u>: $D^{perf}(A) \ni Y$

$$\operatorname{supp} Y = \{p \in \operatorname{Spec} A \mid Y \otimes^{L}_{A} k(p) \neq 0\}$$

<u>Want</u>: realize primes in X in RepTheo way so that we can specialize modules.

<u>III 1. π-points</u>

<u>Def</u>: G fin grp sch. A <u>π-point</u> of G is flat map of algebras

$$\alpha : K[t]/t^p \longrightarrow KG \qquad K/k$$

s.t. it factors through unipotent abelian subgroup scheme.

$$\begin{tikzcd} K[t]/t^p \arrow[rr, "\alpha"] \arrow[dr] & & KG \\ & KU \arrow[ur, "\text{map of Hopf algebras}"'] & \end{tikzcd}$$

where U not nec. def. over k

Examples: $G = E$

$\langle x_d + 1 \rangle$ cyclic shifted subgroup then

$$\begin{array}{ccc} k\langle x_d+1\rangle & \hookrightarrow & kE \\ \wr\! \| & & \\ k[t]/t^p & & \end{array}$$

is π-point

unipotent ab.

$\mathfrak{g}$: $X \in N_p(\mathfrak{g})$

$$\begin{array}{ccc} \mathcal{U}(x) & \hookrightarrow & \mathcal{U}(\mathfrak{g}) \\ \wr\! \| & & \\ k[x]/x^p & & \end{array}$$

univ. envelop

is π-point

unip. ab.

What is the point of a π-point?

π-points will serve as "residue fields"

$\alpha: K[t]/t^p \longrightarrow KG$ induces

$$H^*(\alpha): H^*(G,k) \xrightarrow{-\otimes_k K} H^*(G_K, K) \xrightarrow{\alpha^*} H^*(K[t]/t^p, K)$$

$$H^*(K[t]/t^p, K) \cong K[y] \otimes \Lambda(s)$$

Theorem (Friedlander-P '07)

G fin grp sch. $\forall\, p \in \operatorname{Proj} H^*(G,k)$ $\exists$ π-point $\alpha_p: K[t]/t^p \longrightarrow KG$ for some K/k s.t.

$\sqrt{\ker H^*(\alpha)} = p$.

K/k field ext. then $M_K = M \otimes_k K$
and $M^K = \mathrm{Hom}_k(K, M)$ coextension

M_K & M^K are G_K-modules

Def: π-(co)support of $M \in \mathrm{Rep}\, G$ is def. as

$X \supset \pi\text{-supp}\, M = \{ p \in X \mid \alpha_p^* M_K \text{ not free as } K[t]/t^p\text{-module} \}$

$\pi\text{-cosupp}\, M = \{ p \in X \mid \alpha_p^* M^K \text{ not free as } K[t]/t^p\text{-mod} \}$

see previous table *

Properties	π-supp
1	✓
2	✓
3	✓
4	
5	✓
6	✓

Thm (Friedlander P '07)
BIK P '15

1. $\otimes$ product prop:

$$\pi\text{-supp}(M \otimes N) = \pi\text{-supp}(M) \cap \pi\text{-supp}(N)$$

2. Hom cosupp prop:

$$\pi\text{-cosupp}(\mathrm{Hom}_k(M,N)) = \pi\text{-supp}(M) \cap \pi\text{-cosupp}(N)$$

III 2. Detection of projectivity for π-supp & π-cosupp

{detection for π-supp / π-cosupp} $\overset{\text{formal}}{\Longrightarrow}$ {supp_{BIK} = π-supp / π-cosupp} $\Longrightarrow$ {$\otimes$ product formula / Hom formula for (co)supp_{BIK}}

$\Downarrow$ formal localizing

{Classification of thick tensor ideals in stmod G / StMod G}

cf. Benson-Carlson-Rickard

History: Thm (Chouinard): G f grp
M G-Mod then
M proj $\Leftrightarrow$ $M\downarrow_E$ is proj $\forall E \subset G$ elem. ab.

Thm (Dade 78) $G = E$, M fin dim E-mod
M free $\Leftrightarrow$ $M\downarrow_{\langle x_{\underline{\alpha}}+1\rangle}$ free $\forall \underline{\alpha} \in \mathbb{P}^{n-1}$

Theorem G fin grp sch., $M \in G$-Mod.

M proj $\iff$ $\forall$ K/k $\forall$ π-point $\alpha : K[t]/t^p \to KG$

$\alpha^*(M_K)$ is free.

Corollary: M is proj $\iff$ π-$\mathrm{supp}(M) = \emptyset$

Credit: • $G = E$, M ∞-dim BCR '79

• G unip. Bendel

• $\begin{cases} G \text{ infinitess.} \\ G \text{ any fgs} \end{cases}$ BIKP '15

Corollary: G fin grp sch.

$$\mathrm{supp}_{BIK}(M) = \pi\text{-}\mathrm{supp}(M)$$

Key pt: π-$\mathrm{supp}\, \Gamma_p k = p$

Notation: $\mathrm{supp} = \mathrm{supp}_{BIK} = \pi\text{-}\mathrm{supp}$

Corollary: supp satisfies $\otimes$-product prop

III.3 Classification for StMod$G$

Theorem (BIKP 15) G fgs

$$\left\{\begin{array}{c}\text{localizing}\\ \otimes\text{-ideals}\\ \text{in } \mathrm{StMod}G\end{array}\right\} \underset{\longleftarrow}{\overset{\longrightarrow}{1:1}} \left\{\begin{array}{c}\text{subsets in}\\ \mathrm{Proj}\, H^*(G,k)\end{array}\right\}$$

$$\mathcal{C} \longmapsto \bigcup_{M \in \mathcal{C}} \mathrm{supp}(M)$$

$$\mathcal{C} = \{M \mid \mathrm{supp}(M) \in Z\} \longleftarrow\!\shortmid\; Z$$

and

$$\left\{\begin{array}{c}\text{thick } \otimes\text{-}\\ \text{ideals}\\ \text{in } \mathrm{stmod}\, G\end{array}\right\} \underset{\longleftarrow}{\overset{\longrightarrow}{1:1}} \left\{\begin{array}{c}\text{specialization-}\\ \text{closed}\\ \text{subsets in } \mathrm{Proj}\, H^*(G,k)\end{array}\right\}$$

"proof"

<u>BIK theory</u>

$\Gamma_p \mathrm{StMod}G \overset{\text{def}}{=} \{M \mid \mathrm{supp}(M) = p\}$

<u>Local to Global principal:</u>

classification of localizing $\otimes$-ideals follows from $\Gamma_p \mathrm{StMod}G$ are minimal.

III 4 Reduction to closed points

① want to show $\Gamma_p \mathrm{StMod}\, G$ minimal $\forall p$

If $p = m \in X$ closed then $\Gamma_p \mathrm{StMod}\, G$ min.
(cosupp formula)

② Reduction $b \in H^d(G, k)$

$\rightsquigarrow \quad k \xrightarrow{b} \Omega^{-d} k$

mapping cone $= k /\!/ b$

$\underline{b} = (b_1, \dots, b_n) \qquad k /\!/ \underline{b} = k/\!/_{b_1} \otimes \cdots \otimes k/\!/_{b_n}$

take $K = k(p)$

$$\begin{array}{ccc} \sqrt{p + (\underline{b})} = m \subset X_K = \mathrm{Proj}\, H^*(G_K, K) & & \Gamma_m (K/\!/\underline{b}) \\ \downarrow & & \downarrow \\ p \in X = \mathrm{Proj}\, H^*(G, k) & & \Gamma_p k \end{array}$$

<u>Theorem</u> (BIKP 16)

$$\Gamma_p k \simeq \Gamma_m (K/\!/\underline{b})|_G$$

<u>Corollary</u>: $\Gamma_p \mathrm{StMod}\, G$ minimal.

$\Rightarrow$ classification for $\mathrm{StMod}\, G$.