15/08/16

Antoine Touz´e: Stabilization and cup products for polynomial representations of GLn(k)

Intro: k char $k = p > 0$

G alg grp / k affine $G = GL_n$ or G classical matrix group

rat rep G = V + $\rho : G \longrightarrow GL(V)$

Q1: cohomological info $Ext^*_G(M,N)$?

Q2: Structure of some modules of particular interest?

Problem: understand $\otimes$?

1. M, N P, Q

$$Ext^*(M,N) \otimes Ext^*(P,Q) \xrightarrow{\cup} Ext^*(M \otimes P, N \otimes Q)$$

$\cup$ far from being inj or surj in general.

2. Structure of $M \otimes N$?

open: $[L_1 \otimes L_2 : L_3] = ?$

open: $V = k^n$ def. rep. of GL_n

$$V^{\otimes d} = \bigoplus_i M_i^{\otimes \alpha_i} \quad \text{compute the } \alpha_i \text{?}$$

③ Steinberg tensor product theorem

$G = GL_n$

- Frobenius twists of reps

$M = (M, \rho)$ rep. of GL_n

$M^{[r]} = (M, \rho^{[r]})$

$$\rho^{[r]} : GL_n \xrightarrow{\mathrm{Frob}^r} GL_n \xrightarrow{\rho} GL(M)$$

$$[a_{ij}] \longmapsto [a_{ij}^{p^r}]$$

L simple $\iff L^{[r]}$ simple

- Simple GL_n-modules

$$\left\{\begin{matrix}\text{simple} \\ GL_n\text{-isos}\end{matrix}\right\} \longleftrightarrow (\mathbb{Z}^n)^+$$

$$= \{\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n\}$$

$$L_\lambda \longleftarrow\!\!\mid \lambda$$

L_λ is polynomial, if all $\lambda_i \geq 0$

(L_λ indexed by a partition)

Remark: in general

$L_\lambda \otimes \det^{\otimes n}$ is polynomial $n >> 0$

Def: $\lambda = (\lambda_1, \dots, \lambda_n)$ p^r-restr. $\Leftrightarrow$ $\lambda_n < p^r$

$\lambda_i - \lambda_{i+1} < p^r \ \forall i$

Thm (Steinberg)

$$L_\mu \otimes L_\lambda^{[r]} \simeq L_{\mu + p^r\lambda}$$

μ is p^r-restricted partition.

Thm (T.) $G = GL_n$

M, N, P, Q polynomial reps, $n >> 0$

- $\mathrm{Ext}^*_G(M,N) \otimes \mathrm{Ext}^*_G(P,Q) \xrightarrow{\cup} \mathrm{Ext}^*(M \otimes P, N \otimes Q)$ (+)

  is injectiv

- If $P = P'^{[r]}$, $Q = Q'^{[r]}$ then (+) is iso in low degrees.

Corollary: under same cond. on M

str. of $M \otimes N^{[r]}$ as GL_n-mod is

same as str. of $M \otimes N$ as $GL_n \times GL_n$-mod

Remarks: 1) Cor. gen. Steinberg tensor prod. thm.

2) Thm has analogues for all other classical types.

3) low degrees = explicit bound, dep. on const. $p(M,r)$, $i(M,r)$

these are related to other problems of alg grps

$$\mathcal{P}ol_{d,Gl_n} \xrightarrow{\text{Schur}} \mathfrak{S}_d\text{-modules}$$

$$\mathrm{Ext}^i_{Gl_n}(M,N) \xrightarrow{\text{Schur}} \mathrm{Ext}^i_{\mathfrak{S}_d}(\mathrm{Schur}\,M, \mathrm{Schur}\,N)$$

II Strict poly functions

$$\mathcal{F}_k = \text{cat of functors } F : \{\text{fin dim vec.sp}/k\} \to \{\text{vect sp.}/k\}$$

Def: $F \in \mathcal{F}_k$ strict polynomial of deg d (homog.)

if $\mathrm{Hom}_k(V,W) \xrightarrow{F_{V,W}} \mathrm{Hom}_k(F(V),F(W))$ are all polys of deg d (homog.)

Ex: $\left.\begin{array}{l} S^d : V \longmapsto S^d(V) \\ \Lambda^d : V \longmapsto \Lambda^d(V) \end{array}\right\}$ homog. of deg. d

$\mathcal{P} \subseteq \mathcal{F}_k$ full

obj = strict poly functors

$\mathcal{P}$ is abelian & enoug proj & inj

$\mathcal{P}_d \subset \mathcal{P}$ homog of deg d

$\mathcal{P}_d \cong S(n,d)\text{-Mod}$ $\quad n \geq d$

Schur algebra

Links with GL_n-Mod

$V = k^n$

1. $F \in \mathcal{P}$ then $F(V)$ rat. rep. of GL_n (g acts as $F(g)$)

Reps of this form are called <u>polynomial</u>

Examples:

1) Simple obj of $\mathcal{P}$ are L_λ, λ partition

$$L_\lambda(V) = \begin{cases} L_\lambda & \text{if } k \leq n \\ 0 & \text{if } k > n \end{cases}$$

2) $$I^{(r)} : V \longmapsto \underset{\{v^{p^r} \mid v \in V\}}{\underset{\parallel}{V^{(r)}}} \subseteq S^{p^r}(V)$$

$$F^{(r)} = F \circ I^{(r)}$$

then $F^{(r)}(V) = (F(V))^{(r)}$ as $GL(V)$-mod.

Theorem [FS 1997]

$$\mathrm{Ext}^*_{\mathcal{P}}(F, G) \xrightarrow{\sim} \mathrm{Ext}^*_{GL_n}(F(V), G(V))$$

$n \geq \deg F, \deg G$