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Birational sequences and the tropical Grassmannian

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ABSTRACT

We introduce iterated sequences for Grassmannians, a new class of Fang–Fourier–Littelmann’s birational sequences and explain how they give rise to points in the tropical Grassmannian. For $\text{Gr}(2, \mathbb{C}^n)$ we show that the associated valuations induce toric degenerations. We describe recursively the vertices of the corresponding Newton–Okounkov polytopes, which are particular vertices of a hypercube and hence integral. We show further that every toric degenerations of $\text{Gr}(2, \mathbb{C}^n)$ constructed using the tropical Grassmannian due to Speyer and Sturmfels can be recovered by iterated sequences.

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1. Introduction

Toric degenerations of Grassmannians or more generally flag and spherical varieties have been a vivid research topic during the last 20 years. A novel framework, called *birational sequences*, stemming from representation theory was given by Fang, Fourier and Littelmann in [13]. Within this framework, we introduce and analyze new birational sequences for Grassmannians and discover surprising connections to the tropical Grassmannian by Speyer–Sturmfels [21].

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To be more precise we need to introduce a bit of notation. A *toric degeneration* of a projective variety X is a flat morphism $\pi : \mathcal{X} \rightarrow \mathbb{A}^d$ with generic fiber $\pi^{-1}(t)$ for $t \neq 0$ isomorphic to X and $\pi^{-1}(0)$ a projective toric variety. Consider SL_n over \mathbb{C} with Borel subgroup B of upper triangular matrices. Let P_k be the parabolic subgroup of block-upper triangular matrices with blocks of size $k \times k$ and $(n-k) \times (n-k)$. In particular, the Grassmannian $\text{Gr}(k, \mathbb{C}^n)$ is isomorphic to SL_n/P_k . Denote its dimension by d . Further, let U^- be lower triangular matrices with 1's along the diagonal. The root system of type A_{n-1} is $R = \{\varepsilon_i - \varepsilon_j\}_{i \neq j}$ where $\{\varepsilon_i\}_i$ is the standard basis of \mathbb{R}^n . For every positive root $\beta \in R^+$ we have the one-parameter root subgroup $U_{-\beta} \subset U^-$. In this setting the notion of birational sequences due to Fang, Fourier and Littelmann [13] applies: a sequence $S = (\beta_1, \dots, \beta_d)$ of positive roots is called *birational* for $\text{Gr}(k, \mathbb{C}^n)$ if the multiplication map

$$U_{-\beta_1} \times \dots \times U_{-\beta_d} \rightarrow U^-$$

has image birational to $\text{Gr}(k, \mathbb{C}^n)$. Birational sequences give rise to coordinates on the Grassmannian. These can be used to define valuations on the homogeneous coordinate ring. The construction allows to compute certain values of the valuations explicitly via representations of the Lie algebra \mathfrak{sl}_n . If the valuation has finitely generated value semi-group it induces a toric degeneration of the Grassmannian by [2]. Their setting unifies many known constructions of toric degenerations in representation theory, such as [14, 9, 1, 12]. However, besides the previously known cases few *new* birational sequences were discovered so far. A central question is whether toric degenerations constructed from tropical geometry (see e.g. [21] for the case of Grassmannians) or cluster algebras (see e.g. [20]) can be recovered using birational sequences.

In this paper we study birational sequences that do not correspond to the previously known ones. One way to construct them is provided by the following central Lemma (see also Lemma 2). It allows to obtain a birational sequences for $\text{Gr}(k, \mathbb{C}^{n+1})$ from a birational sequence for $\text{Gr}(k, \mathbb{C}^n)$.

Lemma. *Let $S = (\beta_1, \dots, \beta_d)$ be a birational sequence for $\text{Gr}(k, \mathbb{C}^n)$ and choose i_1, \dots, i_k pairwise differently from $\{1, \dots, n\}$. Then the following is a birational sequence for $\text{Gr}(k, \mathbb{C}^{n+1})$:*

$$S' = (\varepsilon_{i_1} - \varepsilon_{n+1}, \dots, \varepsilon_{i_k} - \varepsilon_{n+1}, \beta_1, \dots, \beta_d).$$

Inspired by the Lemma we define a new class of birational sequences: a birational sequence S for $\text{Gr}(k, \mathbb{C}^n)$ constructed from a birational sequence for $\text{Gr}(k, \mathbb{C}^{k+1})$ by applying the Lemma $n - k - 1$ many times is called *iterated* (see Definition 11).

Focusing on iterated sequences for $\text{Gr}(2, \mathbb{C}^n)$, we show further that the corresponding valuations have the necessary property to induce toric degenerations (relying on a result of [8]). Fixing the Plücker embedding of $\text{Gr}(2, \mathbb{C}^n)$ the corresponding Newton–Okounkov

polytopes are in fact integral polytopes inside the hypercube² in $\mathbb{R}_{\geq 0}^{2(n-2)}$ (see Corollary 2). The vertices are given by the images of the valuation on Plücker coordinates, which in this case form a Khovanskii basis.

Moreover, we identify iterated sequences with maximal cones in $\text{trop}(\text{Gr}(k, \mathbb{C}^n))$, the tropical Grassmannian [21]. For $k = 2$ we give the explicit Algorithm 1 to do so. For an iterated sequence S denote by $C_S \subset \text{trop}(\text{Gr}(2, \mathbb{C}^n))$ the output of the algorithm. We say two toric degenerations of a projective variety X are *equivalent* if their special fibers are isomorphic toric varieties. Our main results (Theorems 1&2) can be summarized as follows:

Theorem. *Let S be an iterated sequence for $\text{Gr}(k, \mathbb{C}^n)$. Then there exists a maximal cone $C \subset \text{trop}(\text{Gr}(k, \mathbb{C}^n))$ such that the toric degeneration of $\text{Gr}(k, \mathbb{C}^n)$ induced by S is equivalent to the one given by the maximal cone $C \subset \text{trop}(\text{Gr}(k, \mathbb{C}^n))$.*

Moreover, for $k = 2$ the cone C equals the cone C_S which is the output Algorithm 1. Conversely, for every maximal cone $C \subset \text{trop}(\text{Gr}(2, \mathbb{C}^n))$ there exists an iterated sequence S such that the induced toric degenerations of C and S are equivalent.

Remark 1. Combining with the results of [3], where toric degenerations of $\text{Gr}(2, \mathbb{C}^n)$ from plabic graphs are studied, our main result implies that iterated sequences not only recover all possible toric degenerations of $\text{Gr}(2, \mathbb{C}^n)$ constructed using the tropical Grassmannian, but also all those constructed using plabic graphs (as in [20]). Said differently, up to equivalence of toric degenerations the following sets are in one-to-one correspondence:

$$\left\{ \begin{array}{l} \text{toric degenerations} \\ \text{of } \text{Gr}(2, \mathbb{C}^n) \text{ from} \\ \text{iterated sequences} \end{array} \right\} \xleftrightarrow{\text{Theorems 1\&2}} \left\{ \begin{array}{l} \text{toric degenerations} \\ \text{of } \text{Gr}(2, \mathbb{C}^n) \text{ from} \\ \text{trop}(\text{Gr}(2, \mathbb{C}^n)) \end{array} \right\} \xleftrightarrow{[3]} \left\{ \begin{array}{l} \text{toric degenerations} \\ \text{of } \text{Gr}(2, \mathbb{C}^n) \text{ from} \\ \text{plabic graphs} \end{array} \right\}$$

We analyze the combinatorial structure of iterated sequences in terms of trivalent trees with n leaves that are in correspondence with maximal cones of $\text{trop}(\text{Gr}(2, \mathbb{C}^n))$. Let \mathcal{T} be the infinite graph with vertices corresponding to (unlabelled) trivalent trees. Two vertices T_1, T_2 of \mathcal{T} are connected by a directed edge $T_1 \rightarrow T_2$, if T_2 can be obtained from T_1 by adding a leaf-edge (see Fig. 1). The unique source of \mathcal{T} is the trivalent tree with three leaves. Algorithm 1 yields the following corollary:

Corollary. *Every iterated sequence S for $\text{Gr}(2, \mathbb{C}^n)$ yields a unique path from the source of \mathcal{T} to the trivalent tree with n leaves corresponding to the equivalence class³ of the maximal cone $C_S \subset \text{trop}(\text{Gr}(2, \mathbb{C}^n))$.*

The paper is structured as follows: after setting up the notation in §2 we recall necessary notions on valuations in §2.1. Reminders on tropical geometry can be found in

² We consider the hypercube as the convex hull of all elements in $\{0, 1\}^{2(n-2)} \subset \mathbb{R}_{\geq 0}^{2(n-2)}$.

³ Equivalence classes are considered with respect to the action of the symmetric group S_n on the Plücker coordinates, see §2.2 below Remark 2.

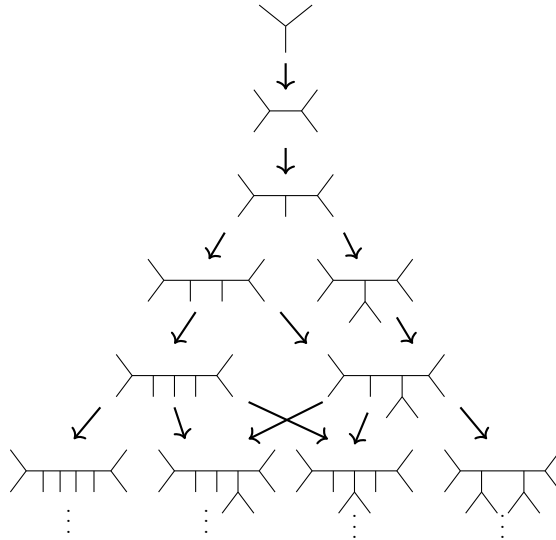


Fig. 1. The tree graph \mathcal{T} from level (# of leaves) 3 to 8.

§2.2. In §3 we introduce iterated (birational) sequences for Grassmannians and recall the construction of the associated valuations due to [13]. In §4 we focus on iterated sequences for $\text{Gr}(2, \mathbb{C}^n)$ and prove the main result.

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2. Notation

Let $I_{k,n}$ be the set of ordered integer sequences $\underline{i} = (i_1 < \dots < i_k)$ with $1 \leq i_1$ and $i_k \leq n$. We consider the Grassmannian $\text{Gr}(k, \mathbb{C}^n)$ of k -dimensional subspaces in \mathbb{C}^n with its Plücker embedding. It is the vanishing of the *Plücker ideal* $\mathcal{I}_{k,n} \subset \mathbb{C}[p_{\underline{j}} \mid \underline{j} \in I_{k,n}]$ in the polynomial ring on *Plücker variables* $p_{\underline{j}}$. To describe $\mathcal{I}_{k,n}$ more explicitly we need some more notation.

Denote $\{1, \dots, n\}$ by $[n]$. Consider $\underline{j} = (j_1 < \dots < j_{k+1}) \in I_{k+1,n}$ and $j_s \in \underline{j}$, i.e. $1 \leq s \leq k+1$. Then $\underline{j} \setminus j_s$ denotes the sequence in $I_{k,n}$ obtained by removing j_s from \underline{j} . For $\underline{i} = (i_1 < \dots < i_{k-1}) \in I_{k-1,n}$ and $j \in [n]$ define

$$\underline{i} \cup j := (i_1 < \dots < i_r < j < i_{r+1} < \dots < i_{k-1}).$$

Set $\ell(\underline{i}, j) := k - (r + 1)$. Note that if $j = i_l$ for some $1 \leq l \leq k - 1$ then $\underline{i} \cup j \notin I_{k,n}$, in this case we set $p_{\underline{i} \cup j} = 0$. The Plücker ideal $\mathcal{I}_{k,n}$ is generated by elements of form

$$R_{\underline{i}, \underline{j}} := \sum_{s=1}^{k+1} (-1)^{\ell(\underline{i}, \underline{j}_s)} p_{\underline{i} \cup \underline{j}_s} p_{\underline{j} \setminus \underline{j}_s}, \tag{2.1}$$

for $\underline{i} \in I_{k-1, n}$ and $\underline{j} \in I_{k+1, n}$ (see e.g. [17, §4.2.2]). For $k = 2$ we simplify notation by $p_{ij} := p_{(i < j)}$ for $(i < j) \in I_{2, n}$. Then the Plücker relations are of form:

$$R_{r, s, u, v} := R_{(r), (s < u < v)} = p_{rs} p_{uv} - p_{ru} p_{sv} + p_{rv} p_{su} \in \mathbb{C}[p_{ij}]_{i < j},$$

with $1 \leq r < s < u < v \leq n$. Let $A_{k, n} := \mathbb{C}[p_{\underline{j}}]_{\underline{j} \in I_{k, n}} / \mathcal{I}_{k, n} \cong \mathbb{C}[\text{Gr}(k, \mathbb{C}^k)]$, which is the homogeneous coordinate ring of the Grassmannian. The cosets $\bar{p}_{\underline{j}} \in A_{k, n}$ are called *Plücker coordinates*.

We recall some standard facts about Lie algebras, algebraic groups and their representations that will be used throughout the rest of the paper. Let \mathfrak{sl}_n be the Lie algebra corresponding to SL_n and fix a Cartan decomposition $\mathfrak{sl}_n = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. The Cartan subalgebra \mathfrak{h} consists of diagonal traceless matrices, and \mathfrak{n}^- (resp. \mathfrak{n}^+) of strictly lower (resp. upper) triangular matrices.

In the type A_{n-1} root system $R = \{\varepsilon_i - \varepsilon_j\}_{i \neq j \in [n]} \subset \mathbb{R}^n$ we have positive roots $R^+ = \{\varepsilon_i - \varepsilon_j\}_{i < j}$ and simple roots $\{\varepsilon_i - \varepsilon_{i+1}\}_{i=1, \dots, n-1}$. Further, set $R_k^+ := \{\varepsilon_i - \varepsilon_j \mid 1 \leq i \leq k \leq j \leq n\}$. For every positive root $\varepsilon_i - \varepsilon_j$ we have a root operator $f_{i, j} \in \mathfrak{n}^-$: it is the elementary matrix with only non-zero entry being 1 in the (j, i) -position.

Let Λ denote the weight lattice, generated by fundamental weights $\omega_1, \dots, \omega_{n-1}$. The k th fundamental representation of \mathfrak{sl}_n is $V(\omega_k) = \bigwedge^k \mathbb{C}^n$. It is cyclically generated by a highest weight vector v_{ω_k} over the universal enveloping algebra $U(\mathfrak{n}^-)$.

Example 1. For $V(\omega_2)$ fix the basis $\{e_i \wedge e_j \mid 1 \leq i < j \leq n\}$. Then the action of \mathfrak{n}^- is given by

$$f_{i, j} \cdot (e_k \wedge e_l) = f_{i, j} \cdot e_k \wedge e_l + e_k \wedge f_{i, j} \cdot e_l = \begin{cases} e_{j+1} \wedge e_l, & \text{if } k = i, \\ e_k \wedge e_{j+1}, & \text{if } l = i, \\ 0, & \text{otherwise.} \end{cases}$$

We choose $e_1 \wedge e_2$ as the highest weight vector v_{ω_2} .

Identifying the Grassmannian with the quotient SL_n/P_k , the Plücker embedding can be written as $\text{Gr}(k, \mathbb{C}^n) \hookrightarrow \mathbb{P}(V(\omega_k))$. For $r \geq 1$ let $V(r\omega_k)^*$ denote the vector space dual to $V(r\omega_k)$. We obtain $A_{k, n} \cong \bigoplus_{r \geq 0} V(r\omega_k)^*$ and the Plücker coordinate $\bar{p}_{\underline{j}} \in A_{k, n}$ for $\underline{j} = (j_1 < \dots < j_k) \in I_{k, n}$ is the dual basis element $(e_{j_1} \wedge \dots \wedge e_{j_k})^* \in V(\omega_k)^*$.

Let $U_k^- \subset U^-$ be the subgroup generated by elements $\exp(z f_{i, j}) = \mathbb{1} + z f_{i, j}$, where $\varepsilon_i - \varepsilon_j \in R_k^+$. Then U_k^- is open and dense in $SL_n/P_k \cong \text{Gr}(k, \mathbb{C}^n)$. Hence, we have Plücker coordinates on U_k^- : $\bar{p}_{\underline{j}}$ is the $k \times k$ -minor on columns $[k]$ and rows \underline{j} . Moreover, we have an isomorphism of the fields of rational functions $\mathbb{C}(\text{Gr}(k, \mathbb{C}^n)) \cong \mathbb{C}(U_k^-)$.

2.1. Valuations

We recall basic notions on valuations and Newton-Okounkov polytopes as presented in [15].

Consider $A = \mathbb{C}[x_1, \dots, x_n]/I$ with I a homogeneous ideal and kernel of $\pi : \mathbb{C}[x_1, \dots, x_n] \rightarrow A$. Denote $\bar{x}_i := \pi(x_i)$. The standard grading on the polynomial ring induces a positive grading on $A = \bigoplus_{i \geq 0} A_i$. Let d be the Krull-dimension of A . Additionally, fix a linear order \prec on the additive abelian group \mathbb{Z}^d .

Definition 1. A map $\mathfrak{v} : A \setminus \{0\} \rightarrow (\mathbb{Z}^d, \prec)$ is a *valuation*, if it satisfies for all $f, g \in A \setminus \{0\}$ and $c \in \mathbb{C}^*$

- (i) $\mathfrak{v}(f + g) \succeq \min\{\mathfrak{v}(f), \mathfrak{v}(g)\}$,
- (ii) $\mathfrak{v}(fg) = \mathfrak{v}(f) + \mathfrak{v}(g)$ and
- (iii) $\mathfrak{v}(cf) = \mathfrak{v}(f)$.

Let $\mathfrak{v} : A \setminus \{0\} \rightarrow (\mathbb{Z}^d, \prec)$ be a valuation. The image $\{\mathfrak{v}(f) \mid f \in A \setminus \{0\}\} \subset \mathbb{Z}^d$ forms an additive semi-group. We denote it by $S(A, \mathfrak{v})$ and refer to it as the *value semi-group*. The *rank* of the valuation is the rank of the sublattice generated by $S(A, \mathfrak{v})$ in \mathbb{Z}^d . We are interested in valuations of *full rank*, i.e. $\text{rank}(\mathfrak{v}) = d$.

One naturally defines a \mathbb{Z}^d -filtration on A by $F_{\mathfrak{v} \succeq a} := \{f \in A \setminus \{0\} \mid \mathfrak{v}(f) \succeq a\} \cup \{0\}$ (and similarly $F_{\mathfrak{v} \succ a}$). The associated graded algebra is

$$\text{gr}_{\mathfrak{v}}(A) := \bigoplus_{a \in \mathbb{Z}^d} F_{\mathfrak{v} \succeq a} / F_{\mathfrak{v} \succ a}. \quad (2.2)$$

If the filtered components $F_{\mathfrak{v} \succeq a} / F_{\mathfrak{v} \succ a}$ are at most one-dimensional for all $a \in \mathbb{Z}^d$, we say \mathfrak{v} has *one-dimensional leaves*. By [16, Theorem 2.3] the full-rank assumption on \mathfrak{v} implies that \mathfrak{v} has one-dimensional leaves. In particular, by [4, Remark 4.13] we then have an isomorphism $\mathbb{C}[S(A, \mathfrak{v})] \cong \text{gr}_{\mathfrak{v}}(A)$. To define a $\mathbb{Z}_{\geq 0}$ -filtration on A induced by \mathfrak{v} we make use of the following lemma:

Lemma. ([9, Lemma 3.2]) *Let S be a finite subset of \mathbb{Z}^d . Then there exists a linear form $e : \mathbb{Z}^d \rightarrow \mathbb{Z}_{\geq 0}$ such that for all $\mathbf{m}, \mathbf{n} \in S$ we have $\mathbf{m} \prec \mathbf{n} \Rightarrow e(\mathbf{m}) > e(\mathbf{n})$ (note the switch!).*

Assume $S(A, \mathfrak{v})$ is finitely generated, more precisely assume $S(A, \mathfrak{v})$ is generated by $\mathfrak{v}(\bar{x}_1), \dots, \mathfrak{v}(\bar{x}_n)$. In this case $\{\bar{x}_1, \dots, \bar{x}_n\}$ is called a *Khovanskii basis*⁴ for (A, \mathfrak{v}) . Now choose a linear form as in the lemma for S being the set of generators. We construct a $\mathbb{Z}_{\geq 0}$ -filtration on A by $F_{\leq m} := \{f \in A \setminus \{0\} \mid e(\mathfrak{v}(f)) \leq m\} \cup \{0\}$ for $m \in \mathbb{Z}_{\geq 0}$. Define similarly $F_{< m}$. The associated graded algebra satisfies

⁴ This term was introduced in [16] generalizing the notion of SAGBI basis.

$$\text{gr}_{\mathbf{v}}(A) \cong \bigoplus_{m \geq 0} F_{\leq m} / F_{< m}. \tag{2.3}$$

For $f \in A \setminus \{0\}$ denote by \bar{f} its image in the quotient $F_{\leq e(\mathbf{v}(f))} / F_{< e(\mathbf{v}(f))}$, hence $\bar{f} \in \text{gr}_{\mathbf{v}}(A)$. We obtain a family of \mathbb{C} -algebras containing A and $\text{gr}_{\mathbf{v}}(A)$ as fibers (see e.g. [2, Proposition 5.1]) that can be defined as follows:

Definition 2. The *Rees algebra* associated with the valuation \mathbf{v} and the filtration $\{F_{\leq m}\}_m$ is the flat $\mathbb{C}[t]$ -subalgebra of $A[t]$ defined as

$$R_{\mathbf{v},e} := \bigoplus_{m \geq 0} (F_{\leq m})t^m. \tag{2.4}$$

It has the properties that $R_{\mathbf{v},e}/tR_{\mathbf{v},e} \cong \text{gr}_{\mathbf{v}}(A)$ and $R_{\mathbf{v},e}/(1-t)R_{\mathbf{v},e} \cong A$. In particular, it defines a flat family over \mathbb{A}^1 (the coordinate on \mathbb{A}^1 given by t). The generic fiber is isomorphic to $\text{Proj}(A)$ and the special fiber is the toric variety $\text{Proj}(\text{gr}_{\mathbf{v}}(A))$, where Proj is taken with respect to the $\mathbb{Z}_{\geq 0}$ -grading on A .

Introduced by Lazarsfeld-Mustață [18] and Kaveh-Khovanskii [15] we recall the definition of Newton-Okounkov body.

Definition 3. Let $\mathbf{v} : A \setminus \{0\} \rightarrow (\mathbb{Z}^d, <)$ be a valuation of full rank. The *Newton-Okounkov body* is

$$\Delta(A, \mathbf{v}) := \overline{\text{conv}\left(\bigcup_{i > 0} \{\mathbf{v}(f)/i \mid 0 \neq f \in A_i\}\right)} \tag{2.5}$$

Anderson showed in [2] that if $\text{gr}_{\mathbf{v}}(A)$ is finitely generated, then $\Delta(A, \mathbf{v})$ is a rational polytope. Moreover, it is the polytope associated to the normalization of the toric variety $\text{Proj}(\text{gr}_{\mathbf{v}}(A))$.

The aim of [8] is to give a criterion for when a valuation induces toric degeneration. Using Anderson’s result, this translates to giving a criterion for when the corresponding value semi-group is finitely generated. We state the criterion [8, Lemma 3 and Theorem 1] below and use it to prove our main theorem in §4. It uses the notion of initial ideals, that are defined as follows. For $u \in \mathbb{Z}_{\geq 0}^n$ let x^u to denote the monomial $x_1^{u_1} \cdots x_n^{u_n} \in \mathbb{C}[x_1, \dots, x_n]$.

Definition 4. Let $f = \sum a_u x^u \in \mathbb{C}[x_1, \dots, x_n]$ and fix an element $w \in \mathbb{R}^n$ (resp. $M \in \mathbb{Z}^{d \times n}$). Then the *initial form* of f with respect to w (resp. M) is

$$\text{in}_w(f) := \sum_{w \cdot u \text{ minimal}} a_u x^u, \quad \text{resp.} \quad \text{in}_M(f) := \sum_{Mm = \min_{\prec} \{Mu \mid a_u \neq 0\}} a_m x^m. \tag{2.6}$$

Let $I \subset \mathbb{C}[x_1, \dots, x_n]$ be an ideal. Then its *initial ideal* with respect to $w \in \mathbb{R}^n$ (resp. $M \in \mathbb{Z}^{d \times n}$) is defined as

$$\text{in}_w(I) := \langle \text{in}_w(f) \mid f \in I \rangle, \quad \text{resp.} \quad \text{in}_M(I) := \langle \text{in}_M(f) \mid f \in I \rangle.$$

Definition 5. Given a valuation $\mathfrak{v} : A \setminus \{0\} \rightarrow (\mathbb{Z}^d, \prec)$ define the *weighting matrix* of \mathfrak{v} by

$$M_{\mathfrak{v}} := (\mathfrak{v}(\bar{x}_1), \dots, \mathfrak{v}(\bar{x}_n)) \in \mathbb{Z}^{d \times n}.$$

That is, the columns of $M_{\mathfrak{v}}$ are given by the images $\mathfrak{v}(\bar{x}_i)$ for $i \in [n]$.

Theorem. ([8, Theorem 1]) Let $\mathfrak{v} : A \setminus \{0\} \rightarrow (\mathbb{Z}^d, \prec)$ be a full-rank valuation with $M_{\mathfrak{v}} \in \mathbb{Z}^{d \times n}$ the weighting matrix of \mathfrak{v} . Then

$$S(A, \mathfrak{v}) \text{ is generated by } \{\mathfrak{v}(\bar{x}_i)\}_{i \in [n]} \iff \text{in}_{M_{\mathfrak{v}}}(I) \text{ is prime and } M_{\mathfrak{v}} \text{ is of full rank.}$$

In this case $\{\bar{x}_1, \dots, \bar{x}_n\}$ forms a Khovanskii basis for (A, \mathfrak{v}) and $\Delta(A, \mathfrak{v}) = \text{conv}(\mathfrak{v}(\bar{x}_i) \mid i \in [n])$.

In particular, the theorem (resp. its proof) implies that if $\text{in}_{M_{\mathfrak{v}}}(I)$ is prime, we have $\text{gr}_{\mathfrak{v}}(A) \cong \mathbb{C}[x_1, \dots, x_n] / \text{in}_{M_{\mathfrak{v}}}(I)$. To make a connection to tropical geometry, more precisely the tropical Grassmannian, we rely on the following Lemma.

Lemma. ([8, Lemmata 2&3]) Let $\mathfrak{v} : A \setminus \{0\} \rightarrow (\mathbb{Z}^d, \prec)$ be a full-rank valuation and $M_{\mathfrak{v}}$ the associated weighting matrix. Then $\text{in}_{M_{\mathfrak{v}}}(I)$ is monomial-free. Moreover, there exists $w \in \mathbb{Z}^n$ with $\text{in}_w(I) = \text{in}_{M_{\mathfrak{v}}}(I)$.

2.2. Tropical geometry

We recall basic notions on tropical geometry, in particular on the tropical Grassmannian. For details we refer to [19] and [21]. For $u \in \mathbb{Z}^n$ as before we denote $x^u = x_1^{u_1} \cdots x_n^{u_n} \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

Definition 6. Let $f = \sum a_u x^u \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. The *tropicalization* of f is the function $f^{\text{trop}} : \mathbb{R}^n \rightarrow \mathbb{R}$ given by

$$f^{\text{trop}}(w) := \min\{w \cdot u \mid u \in \mathbb{Z}^n \text{ and } a_u \neq 0\}.$$

If $w - v = (m, \dots, m)$, for some $v, w \in \mathbb{R}^n$ and $m \in \mathbb{R}$, we have that the minimum in $f^{\text{trop}}(w)$ and $f^{\text{trop}}(v)$ is achieved for the same $u \in \mathbb{Z}^n$ with $a_u \neq 0$.

Definition 7. ([19, Definitions 3.1.1/2]) Let $f = \sum a_u x^u \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ and $V(f)$ the associated hypersurface in the algebraic torus $T^n = (\mathbb{C}^*)^n$. Then the *tropical hypersurface* of f is

$$\text{trop}(V(f)) := \left\{ w \in \mathbb{R}^n \mid \begin{array}{l} \text{the minimum in } f^{\text{trop}}(w) \\ \text{is achieved at least twice} \end{array} \right\}.$$

Let I be an ideal in $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. The *tropicalization* of the variety $V(I) \subset T^n$ is defined as

$$\text{trop}(V(I)) := \bigcap_{f \in I} \text{trop}(V(f)) \subset \mathbb{R}^n.$$

For a projective variety $V(I) \subset \mathbb{P}^{n-1}$ with I a homogeneous ideal in $\mathbb{C}[x_1, \dots, x_n]$ we consider the ideal $\hat{I} := IC[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Then $V(\hat{I}) = V(I) \cap T^n$. The tropicalization of a projective variety is defined as $\text{trop}(V(I)) := \text{trop}(V(\hat{I}))$.

Recall the notion of initial ideal from Definition 4 and consider for $w \in \mathbb{R}^n$ the initial ideal $\text{in}_w(I)$. By [11, Theorem 15.17], there exists a flat family over \mathbb{C} whose generic fiber over $t \neq 0$ is isomorphic to $\mathbb{C}[x_1, \dots, x_n]/I$ and whose special fiber over $t = 0$ is isomorphic to $\mathbb{C}[x_1, \dots, x_n]/\text{in}_w(I)$. It is given by the following family of ideals

$$\tilde{I}_t := \left\langle t^{-\min_u \{w \cdot u\}} f(t^{w_1} x_1, \dots, t^{w_n} x_n) \mid f = \sum a_u x^u \in I \right\rangle \subset \mathbb{C}[t, x_1^{\pm 1}, \dots, x_n^{\pm 1}]. \quad (2.7)$$

Let I_s denote the ideal $\tilde{I}_t|_{t=s}$. For $s \neq 0$ the isomorphism $\mathbb{C}[x_1, \dots, x_n]/I_s \cong \mathbb{C}[x_1, \dots, x_n]/I_1 = \mathbb{C}[x_1, \dots, x_n]_i/I$ is given by a ring automorphism of $\mathbb{C}[x_1, \dots, x_n]$ sending I_s to I . If $\text{in}_w(I)$ is *toric*, i.e. a binomial prime ideal, then $V(\text{in}_w(I))$ is a toric variety (see e.g. [19, Lemma 2.4.14]) and flat degeneration of $V(I)$ with family defined by \tilde{I}_t . To find toric initial ideals, it is reasonable to consider the tropicalization of $V(I)$ as due to the *Fundamental Theorem of Tropical Geometry* [19, Theorem 3.2.3] we have

$$\text{trop}(V(I)) = \{w \in \mathbb{R}^n \mid \text{in}_w(I) \text{ is monomial-free}\}.$$

Further, by the *Structure Theorem* [19, Theorem 3.3.5] $\text{trop}(V(I))$ is the support of a pure rational d -dimensional polyhedral fan, where d is the Krull-dimension of I . It contains a linear subspace, called *lineality space* defined as

$$\text{trop}_0(V(I)) := \{w \in \text{trop}(V(I)) \mid \text{in}_w(I) = I\}.$$

If I is homogeneous we have $\mathbb{R}(1, \dots, 1) \subset \text{trop}_0(V(I))$. It is often convenient to consider the quotient of $\text{trop}(V(I))$ by the lineality space. We can choose a fan structure on $\text{trop}(V(I))$ by considering it as a subfan of the Gröbner fan of I . In particular, if v and w lie in the relative interior of a cone C , denoted by $v, w \in C^\circ$, if and only if

$$\text{in}_w(I) = \text{in}_v(I).$$

We therefore adopt the notation $\text{in}_C(I) := \text{in}_w(I)$ for an arbitrary $w \in C^\circ$. We say a cone C is *prime*, if $\text{in}_C(I)$ is a prime ideal.

Definition 8. The *tropical Grassmannian*, denoted $\text{trop}(\text{Gr}(k, \mathbb{C}^n)) \subset \mathbb{R}^{\binom{n}{k}}$ is the tropical variety of the Plücker ideal $\mathcal{I}_{k,n}$. By [21, Corollary 3.1] it is a $k(n-k)+1$ -dimensional polyhedral fan whose maximal cones are all of this dimension.

We mainly focus on the tropicalization of $\text{Gr}(2, \mathbb{C}^n)$. By [21, Corollary 4.4] $\text{trop}(\text{Gr}(2, \mathbb{C}^n))$ has the very nice property that every initial ideal $\text{in}_C(\mathcal{I}_{2,n})$ associated to a maximal cone $C \subset \text{trop}(\text{Gr}(2, \mathbb{C}^n))$ is toric. Further, Speyer and Sturmfels show the following:

Theorem. ([21, Theorem 3.4]) *The quotient $\text{trop}(\text{Gr}(2, \mathbb{C}^n)) / \text{trop}_0(\text{Gr}(2, \mathbb{C}^n)) \subset \mathbb{R}^{\binom{n}{2}} / \mathbb{R}^n$ intersected with the unit sphere is, up to sign, the space of phylogenetic trees [5].*

The theorem implies that every maximal prime cone C can be associated with a *labeled trivalent tree* with n leaves. The set of all labeled trivalent trees with n leaves is denoted by \mathcal{T}_n . A *trivalent tree* is a graph with internal vertices of valency three and no loops or cycles of any kind. Non-internal vertices are called *leaves* and the word *labeled* refers to labeling the leaves by $1, \dots, n$. We call an edge *internal*, if it connects two internal vertices.

We label the standard basis of $\mathbb{R}^{\binom{n}{2}}$ by ordered sequences $(i < j) \in I_{2,n}$ corresponding to Plücker variables. The following definition shows how to obtain a point in the relative interior of a maximal cone in $\text{trop}(\text{Gr}(2, \mathbb{C}^n))$ from a labeled trivalent tree. It follows from [21, Theorem 3.4].

Definition 9. Let T be a labeled trivalent tree with n leaves. Then the $(i < j)$ 'th entry of the weight vector $w_T \in \text{trop}(\text{Gr}(2, \mathbb{C}^n))$ is

$$-\#\{\text{internal edges on path from leaf } i \text{ to leaf } j \text{ in } T\}.$$

For notational convenience we set $\text{in}_T(\mathcal{I}_{2,n}) := \text{in}_{w_T}(\mathcal{I}_{2,n})$. The corresponding maximal cone in $\text{trop}(\text{Gr}(2, \mathbb{C}^n))$ is denoted C_T .

Remark 2. Combining the above, we have that every trivalent labeled tree induces a toric degeneration of $\text{Gr}(2, \mathbb{C}^n)$ with flat family as given in (2.7). A main result of [16] is that there is a full-rank valuation with finitely generated value semi-group associated to prime cones in the tropicalization of a projective variety. This yields a flat family using Rees algebras as in Definition 2.

The symmetric group S_n acts on \mathcal{T}_n by permuting the labels of the leaves of a tree. We also have an S_n -action on Plücker coordinates given by

$$\sigma(p_{ij}) = \text{sgn}(\sigma)p_{\sigma^{-1}(i)\sigma^{-1}(j)} \text{ for } \sigma \in S_n.$$

If $\sigma^{-1}(i) > \sigma^{-1}(j)$, we set $p_{\sigma^{-1}(i)\sigma^{-1}(j)} := -p_{\sigma^{-1}(j)\sigma^{-1}(i)}$. The S_n -action induces a ring automorphism of $\mathbb{C}[p_{ij}]_{i < j}$ for every $\sigma \in S_n$. It sends $\text{in}_T(\mathcal{I}_{2,n})$ to $\text{in}_{\sigma(T)}(\mathcal{I}_{2,n})$ for every trivalent labeled tree $T \in \mathcal{T}_n$. Denote the equivalence class of T with respect to this action by T . It is uniquely determined by the underlying (*unlabeled*) *trivalent tree* with n leaves, see for example Fig. 2. We denote the set of trivalent trees by \mathcal{T}_n/S_n .

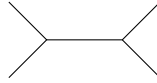


Fig. 2. A trivalent tree with 4 leaves.

Consider a trivalent tree $T \in \mathcal{T}_n/S_n$. If there are two non-internal edges connected to the same internal vertex c , then we say T has a *cherry* at vertex c .

Lemma 1. *Every trivalent tree with $n \geq 4$ leaves has a cherry.*

Proof. We use induction on n . For $n = 4$ Fig. 2 displays the only trivalent tree in \mathcal{T}_4/S_4 and we see, it has two cherries. Now consider a trivalent tree $T' \in \mathcal{T}_{n+1}/S_{n+1}$. We remove one edge connected to a leaf and obtain a tree $T \in \mathcal{T}_n/S_n$. By induction, T has a cherry at some vertex c . Adding the removed edge back there are two possibilities: either we add it to an internal edge (creating a new internal vertex), then the cherry also exists in T' . Or we add it at an edge with a leaf, hence create a new cherry. \square

3. Birational sequences

We recall the definition of birational sequences due to Fang, Fourier, and Littelmann in [13] and associated valuations. After proving the central Lemma 2 we define a new class of birational sequences called *iterated* in Definition 11.

From now on let $d := k(n - k)$ be the dimension of $\text{Gr}(k, \mathbb{C}^n)$. Consider a positive root $\beta \in R^+$. The *root subgroup* corresponding to β is given by $U_{-\beta} := \{\exp(zf_\beta) \mid z \in \mathbb{C}\} \subset U^-$, where $\exp(zf_\beta) = \mathbb{1} + zf_\beta$.

Definition 10. ([13, Definition 2]) Let $S = (\beta_1, \dots, \beta_d)$ be a sequence of positive roots. Then S is called a *birational sequence* for $\text{Gr}(k, \mathbb{C}^n)$ if the multiplication map

$$\psi_S := \text{mult} : U_{-\beta_1} \times \dots \times U_{-\beta_d} \rightarrow U^-$$

has image birational to U_k^- .

Notice that $U_{-\beta_1} \times \dots \times U_{-\beta_d} \cong \mathbb{A}^d$.

Example 2. The following are two first (and motivating) examples of birational sequences:

1. The multiplication map $\prod_{\beta \in R_k^+} U_{-\beta} \rightarrow U^-$ has image U_k^- and hence, is birational. In particular, every sequence containing all roots in R_k^+ (in arbitrary order) is a birational sequence called *PBW-sequence*, see [13, Example 1 and page 131]. We distinguish between PBW-sequences S and S' when the order of the roots in S is different from the order of the roots in S' .

2. For w_0 the longest element in S_n , let $w_k \in S_n / \langle s_i \mid i \neq k \rangle$ denote a coset representative of w_0 . Fix a reduced decomposition $\underline{w}_k = s_{i_1} \dots s_{i_d}$ of w_k and let $S = (\varepsilon_{i_1} - \varepsilon_{i_1+1}, \dots, \varepsilon_{i_d} - \varepsilon_{i_d+1})$ be the corresponding sequence of simple roots. Then S is a birational sequence, it is referred to as *the reduced decomposition case* in [13, Example 2].

The second example shows that repetitions of positive roots may occur in birational sequences. Our aim is to shed some light on sequences that are *neither* PBW *nor* associated to reduced decompositions for Grassmannians. The following lemma allows us to construct such sequences for $\text{Gr}(k, \mathbb{C}^{n+1})$ from sequences for $\text{Gr}(k, \mathbb{C}^n)$.

Lemma 2. *Let $S = (\beta_1, \dots, \beta_d)$ be a birational sequence for $\text{Gr}(k, \mathbb{C}^n)$ and chose $I = \{i_1, \dots, i_k\} \subset [n]$ with $|I| = k$. Then the following is a birational sequence for $\text{Gr}(k, \mathbb{C}^{n+1})$:*

$$S' := (\varepsilon_{i_1} - \varepsilon_{n+1}, \dots, \varepsilon_{i_k} - \varepsilon_{n+1}, \beta_1, \dots, \beta_d). \tag{3.1}$$

Proof. The elements of $\text{im}(\psi_{S'})$ are of the form

$$\exp(y_1 f_{i_1, n+1}) \cdots \exp(y_k f_{i_k, n+1}) \exp(z_1 f_{\beta_1}) \cdots \exp(z_d f_{\beta_d}) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ a_{2,1} & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & 0 \\ a_{n+1,1} & \dots & a_{n+1,n} & 1 \end{pmatrix},$$

for $y_1, \dots, y_k, z_1, \dots, z_d \in \mathbb{C}$. They satisfy $a_{n+1,j} = y_1 a_{i_1,j} + \dots + y_k a_{i_k,j}$ for $1 \leq i \leq n$. Deleting the last row and column gives the image of ψ_S , i.e. for $1 \leq i < j \leq n$ we have $a_{i,j} \in \mathbb{C}[z_1, \dots, z_d]$. We need to show that $\text{im}(\psi_{S'})$ is birational to $U_k^- \subset SL_{n+1}$ given that $\text{im}(\psi_S)$ is birational to $U_k^- \subset SL_n$. Denote the corresponding map for S by $\phi_S : \text{im}(\psi_S) \dashrightarrow U_k^- \subset SL_n$. Recall that elements of U_k^- are block matrices of form

$$(x_{i,j})_{i,j} = \begin{pmatrix} \mathbb{1} & 0 \\ * & \mathbb{1} \end{pmatrix},$$

where non-trivial $x_{i,j}$ occur for $1 \leq j \leq k$ and $k+1 \leq i \leq n$. Let $\underline{i} \in I_{k,n}$ be the ordered sequence with entries i_1, \dots, i_k . We define a birational map $\phi_{S'} : \text{im}(\psi_{S'}) \dashrightarrow U^- \subset SL_{n+1}$ by specifying for $1 \leq j \leq k$

$$y_j \mapsto \frac{(-1)^{k-j+1} \bar{p}_{\underline{i} \setminus i_j \cup n+1}}{\bar{p}_{\underline{i}}} \quad \text{and} \quad a_{r,s} \mapsto \phi_S(a_{r,s}) = x_{r,s} \quad \text{if} \quad 1 \leq r, s \leq n.$$

Then $\phi_{S'}(a_{n+1,j}) = \sum_{l=1}^k \phi_{S'}(y_l) \phi_S(a_{i_l,j})$. Before continuing with the computation note that on $U_k^- \subset SL_{n+1}$ we have $\bar{p}_{(1 < \dots < k) \setminus j \cup i_l}((x_{r,s})_{r,s}) = x_{i_l,j}$ for $1 \leq l \leq k$. If $j \leq k$, making use of the Plücker relations (2.1) this yields the following:

$$\begin{aligned} \phi_{S'}(a_{n+1,j}) &= \sum_{l=1}^k \phi_{S'}(y_l)\phi_S(a_{i_l,j}) = \sum_{l=1}^k \frac{(-1)^{k-l+1}\bar{p}_{\underline{l}\setminus i_l\cup n+1}}{\bar{p}_{\underline{l}}}x_{i_l,j} \\ &= \sum_{l=1}^k \frac{(-1)^{k-l+1}\bar{p}_{\underline{l}\setminus i_l\cup n+1}}{\bar{p}_{\underline{l}}}\bar{p}_{(1<\dots<k)\setminus j\cup i_l} = \bar{p}_{(1<\dots<k)\setminus j\cup n+1} = x_{n+1,j}. \end{aligned}$$

For $l > k$ we obtain $\phi_{S'}(a_{n+1,l}) \in \mathbb{C}(x_{i,j} | 1 \leq j \leq k, k + 1 \leq i \leq n)$. In particular, we have $\mathbb{C}(\text{im}(\phi_{S'})) \cong \mathbb{C}(U_k^-) \cong \mathbb{C}(\text{Gr}(k, \mathbb{C}^{n+1}))$. By little abuse of notation we denote the invertible maps induced by S between $\mathbb{C}(\text{Gr}(k, \mathbb{C}^n))$ and $\mathbb{C}(\mathbb{A}^d) \cong \mathbb{C}(z_1, \dots, z_d)$ by ψ_S^* and $(\psi_S^*)^{-1}$, respectively. Induced by the pullback of $\psi_{S'}$ we consider the map between the rings of rational functions (also denoted $\psi_{S'}^*$):

$$\begin{aligned} \psi_{S'}^* : \mathbb{C}(\text{Gr}(k, \mathbb{C}^{n+1})) &\rightarrow \mathbb{C}(\mathbb{A}^{d+k}) \cong \mathbb{C}(z_1, \dots, z_d, y_1, \dots, y_k) \\ \bar{p}_{\underline{j}} &\mapsto \begin{cases} \psi_S^*(\bar{p}_{\underline{j}}) & \text{if } \underline{j} \in I_{k,n}, \\ \sum_{l=1}^k y_l \psi_S^*((-1)^{\ell(\underline{j}, i_l)} \bar{p}_{\underline{j}\setminus n+1\cup i_l}) & \text{if } n+1 \in \underline{j}. \end{cases} \end{aligned}$$

A straightforward computation reveals that the following map is indeed inverse to $\psi_{S'}^*$, above

$$\begin{aligned} (\psi_{S'}^*)^{-1} : \mathbb{C}(\mathbb{A}^{d+k}) &\rightarrow \mathbb{C}(\text{Gr}(k, \mathbb{C}^{n+1})) \cong \mathbb{C}(\bar{p}_{\underline{j}})_{\underline{j} \in I_{k,n+1}} \\ z_i &\mapsto (\psi_S^*)^{-1}(z_i) \in \mathbb{C}(\bar{p}_{\underline{j}})_{\underline{j} \in I_{k,n}}, \\ y_j &\mapsto \frac{(-1)^{k-j+1}\bar{p}_{\underline{j}\setminus i_j\cup n+1}}{\bar{p}_{\underline{j}}}, \end{aligned}$$

which completes the proof. \square

Definition 11. For $k < n$ consider a birational sequence for $\text{Gr}(k, \mathbb{C}^{k+1})$. Now extend it as in (3.1) to a birational sequence for $\text{Gr}(k, \mathbb{C}^{k+2})$. Repeat this process until the outcome is a birational sequence for $\text{Gr}(k, \mathbb{C}^n)$. Birational sequences of this form are called *iterated*.

We explain how to obtain a valuation from a fixed birational sequence $S = (\beta_1, \dots, \beta_d)$ for $\text{Gr}(k, \mathbb{C}^n)$ as constructed in [13, §7]. Define the *height function* $\text{ht} : R^+ \rightarrow \mathbb{Z}_{\geq 0}$ by $\text{ht}(\varepsilon_i - \varepsilon_j) := j - i$. Then the *height weighted function* $\Psi_S : \mathbb{Z}^d \rightarrow \mathbb{Z}$ is given by

$$\Psi_S(m_1, \dots, m_d) := \sum_{i=1}^d m_i \text{ht}(\beta_i).$$

Let $<_{\text{lex}}$ be the lexicographic order on \mathbb{Z}^d . The Ψ_S -weighted reverse lexicographic order \prec_{Ψ_S} on \mathbb{Z}^d is defined by setting for $m, n \in \mathbb{Z}^d$

$$m \prec_{\Psi_S} n \Leftrightarrow \Psi_S(m) < \Psi_S(n) \text{ or } \Psi_S(m) = \Psi_S(n) \text{ and } m >_{\text{lex}} n. \tag{3.2}$$

When the sequence S is clear from the context we denote Ψ_S by Ψ .

Definition 12. ([13, §7]) Let $f = \sum a_u y^u$ with $u \in \mathbb{Z}_{\geq 0}^d$ be a non-zero polynomial in $\mathbb{C}[y_1, \dots, y_d]$. The valuation $\mathbf{v}_S : \mathbb{C}[y_1, \dots, y_d] \setminus \{0\} \rightarrow (\mathbb{Z}_{\geq 0}^d, \prec_\Psi)$ associated to S is defined as

$$\mathbf{v}_S(f) := \min_{\prec_\Psi} \{u \in \mathbb{Z}_{\geq 0}^d \mid a_u \neq 0\}. \tag{3.3}$$

We extend \mathbf{v}_S to a valuation on $\mathbb{C}(y_1, \dots, y_d) \setminus \{0\}$ by $\mathbf{v}_S(\frac{f}{g}) := \mathbf{v}_S(f) - \mathbf{v}_S(g)$ for $f, g \in \mathbb{C}[y_1, \dots, y_d] \setminus \{0\}$.

Valuations of form (3.3) are called *lowest term valuations*. As S is a birational sequence, for every $f \in \mathbb{C}(\text{Gr}(k, \mathbb{C}^n))$ there exists a unique element $\psi_S^*(f) \in \mathbb{C}(y_1, \dots, y_d)$. Hence, we have a valuation on $\mathbb{C}(\text{Gr}(k, \mathbb{C}^n)) \setminus \{0\}$ given by $\mathbf{v}_S(f) := \mathbf{v}_S(\psi_S^*(f))$. We restrict to obtain

$$\mathbf{v}_S : A_{k,n} \setminus \{0\} \rightarrow (\mathbb{Z}_{\geq 0}^d, \prec_\Psi).$$

Denote by $S(A_{k,n}, \mathbf{v}_S)$ the associated value semi-group and the associated graded algebra by $\text{gr}_S(A_{k,n})$. For the images of Plücker coordinates $\bar{p}_j \in A_{k,n}$ we choose as before the notation $\bar{p}_j \in \text{gr}_S(A_{k,n})$. The weighting matrix for \mathbf{v}_S is denoted M_S .

Theorem 1. *Let S be an iterated sequence for $\text{Gr}(k, \mathbb{C}^n)$ and $\mathbf{v}_S : A_{k,n} \setminus \{0\} \rightarrow (\mathbb{Z}_{\geq 0}^d, \prec_\Psi)$ the corresponding valuation. Then there exists a cone $C \subset \text{trop}(\text{Gr}(k, \mathbb{C}^n))$ such that*

$$\text{in}_{M_S}(\mathcal{I}_{k,n}) = \text{in}_C(\mathcal{I}_{k,n}).$$

Proof. We need to show that \mathbf{v}_S is a full rank valuation. Then by [8, Corollary 3] the initial ideal $\text{in}_{M_S}(\mathcal{I}_{k,n})$ is monomial-free. Moreover, there exists $w \in \mathbb{Z}^{\binom{n}{k}}$ with $\text{in}_w(\mathcal{I}_{k,n}) = \text{in}_{M_S}(\mathcal{I}_{k,n})$. In particular, we have a cone $C \subset \text{trop}(\text{Gr}(k, \mathbb{C}^n))$ with $w \in C^\circ$.

To prove that \mathbf{v}_S has full rank, it suffices to show that M_S has full rank. As S is an iterated sequence we pursue by induction on n . If $n = k + 1$ we may assume that $S = (\varepsilon_1 - \varepsilon_{k+1}, \dots, \varepsilon_k - \varepsilon_{k+1})$. In this case, after choosing an appropriate order on Plücker coordinates, M_S contains the identity matrix as a submatrix. Now assume that the claim is true for $n - 1$ and let $S = (\varepsilon_{i_1} - \varepsilon_n, \dots, \varepsilon_{i_k} - \varepsilon_n, \beta_1, \dots, \beta_{d'})$, where $(\beta_1, \dots, \beta_{d'})$ is an iterated sequence for $\text{Gr}(k, \mathbb{C}^{n-1})$. We have to show that the rows of M_S corresponding to $\varepsilon_{i_1} - \varepsilon_n, \dots, \varepsilon_{i_k} - \varepsilon_n$ are linearly independent. Consider $\underline{j}^1, \dots, \underline{j}^k \in I_{k,n}$ such that for all $l \leq k$ we have $i_1, \dots, i_{l-1}, n \in \underline{j}^l$ and $i_l \notin \underline{j}^l$. Compute the columns of M_S corresponding to $\mathbf{v}_S(\bar{p}_{\underline{j}^1}), \dots, \mathbf{v}_S(\bar{p}_{\underline{j}^k})$ using [13, Proposition 2]. We see that the submatrix with rows corresponding to $\varepsilon_{i_1} - \varepsilon_n, \dots, \varepsilon_{i_k} - \varepsilon_n$ is a $k \times k$ -identity matrix. Hence, the columns of $\mathbf{v}_S(\bar{p}_{\underline{j}^1}), \dots, \mathbf{v}_S(\bar{p}_{\underline{j}^k})$ are linearly independent and so M_S is of full rank by induction. \square

Corollary 1. *For every iterated sequence S the weighting matrix M_S is of full rank.*

Table 1
 Images of Plücker coordinates under the valuations $\mathbf{v}_S, \mathbf{v}_{S'}$ associated to $S = (\varepsilon_1 - \varepsilon_4, \varepsilon_2 - \varepsilon_4, \varepsilon_1 - \varepsilon_3, \varepsilon_2 - \varepsilon_3)$, respectively $S' = (\varepsilon_3 - \varepsilon_4, \varepsilon_2 - \varepsilon_3, \varepsilon_1 - \varepsilon_3, \varepsilon_2 - \varepsilon_3)$ for $\text{Gr}(2, \mathbb{C}^4)$.

Plücker	\mathbf{v}_S	$\mathbf{v}_{S'}$
\bar{p}_{12}	(0, 0, 0, 0)	(0, 0, 0, 0)
\bar{p}_{13}	(0, 0, 0, 1)	(0, 0, 0, 1)
\bar{p}_{23}	(0, 0, 1, 0)	(0, 0, 1, 0)
\bar{p}_{14}	(0, 1, 0, 0)	(1, 0, 0, 1)
\bar{p}_{24}	(1, 0, 0, 0)	(1, 0, 1, 0)
\bar{p}_{34}	(1, 0, 0, 1)	(0, 1, 1, 0)

4. Iterated sequences for $\text{Gr}(2, \mathbb{C}^n)$

In this subsection we prove Theorem 2 stated in the introduction. After proving Proposition 1 we can apply [8, Theorem 1] to complete the proof. We focus on iterated sequences for $\text{Gr}(2, \mathbb{C}^n)$ and start by computing the images of Plücker coordinates under the associated valuations.

Let $S = (\beta_1, \dots, \beta_d)$ be a birational sequence for $\text{Gr}(2, \mathbb{C}^n)$. Let $U(\mathfrak{n}_S^-) \subset U(\mathfrak{n}^-)$ be the subalgebra generated by monomials of form $f_{\beta_1}^{m_1} \dots f_{\beta_d}^{m_d}$. We consider the irreducible highest weight representation $V(\omega_2) = \bigwedge^2 \mathbb{C}^n = U(\mathfrak{n}_S^-) \cdot (e_1 \wedge e_2)$. There exists at least one monomial of form $\mathbf{f}_S^m = f_{\beta_1}^{m_1} \dots f_{\beta_d}^{m_d}$ with the property $\mathbf{f}_S^m \cdot (e_1 \wedge e_2) = e_i \wedge e_j$ for all $1 \leq i, j \leq n$. Then by [13, Proposition 2] we have

$$\mathbf{v}_S(\bar{p}_{ij}) = \min_{\prec_\Psi} \{m \in \mathbb{Z}_{\geq 0}^d \mid \mathbf{f}_S^m \cdot (e_1 \wedge e_2) = e_i \wedge e_j\}. \tag{4.1}$$

Example 3. Consider $\text{Gr}(2, \mathbb{C}^4)$ and the iterated sequence $S = (\varepsilon_1 - \varepsilon_4, \varepsilon_2 - \varepsilon_4, \varepsilon_1 - \varepsilon_3, \varepsilon_2 - \varepsilon_3)$, respectively $S' = (\varepsilon_3 - \varepsilon_4, \varepsilon_2 - \varepsilon_4, \varepsilon_1 - \varepsilon_3, \varepsilon_2 - \varepsilon_3)$. They are birational by Lemma 2, as $(\varepsilon_1 - \varepsilon_3, \varepsilon_2 - \varepsilon_3)$ is of PBW-type for $\text{Gr}(2, \mathbb{C}^3)$. We compute the valuation \mathbf{v}_S on Plücker coordinates. There are two monomials sending $e_1 \wedge e_2$ to $e_3 \wedge e_4$, namely

$$\mathbf{f}_S^{(1,0,0,1)} \cdot (e_1 \wedge e_2) = \mathbf{f}_S^{(0,1,1,0)} \cdot (e_1 \wedge e_2) = e_1 \wedge e_4.$$

We have $\Psi_S(1, 0, 0, 1) = \Psi_S(0, 1, 1, 0) = 4$, but $(1, 0, 0, 1) >_{lex} (0, 1, 1, 0)$. Hence, $\mathbf{v}_S(\bar{p}_{34}) = (1, 0, 0, 1)$.

For $\mathbf{v}_{S'}$ we compute $\mathbf{f}_S^{(1,0,0,1)}(e_1 \wedge e_2) = \mathbf{f}_S^{(0,1,0,0)}(e_1 \wedge e_2) = e_1 \wedge e_4$. Again, we have $\Psi_{S'}(1, 0, 0, 1) = \Psi_{S'}(0, 1, 0, 0) = 2$, but as $(1, 0, 0, 1) >_{lex} (0, 1, 0, 0)$ it follows $\mathbf{v}_{S'}(\bar{p}_{14}) = (1, 0, 0, 1)$. In Table 1 you can find the images of all Plücker coordinates under \mathbf{v}_S and $\mathbf{v}_{S'}$.

From now on we fix an iterated birational sequence $S = (\varepsilon_{i_n} - \varepsilon_n, \varepsilon_{j_n} - \varepsilon_n, \dots, \varepsilon_{i_3} - \varepsilon_3, \varepsilon_{j_3} - \varepsilon_3)$ for $\text{Gr}(2, \mathbb{C}^n)$, where $i_l, j_l \in [l - 1]$ with $i_l \neq j_l$ for all $3 \leq l \leq n$. Then Algorithm 1 associates to S a labeled trivalent tree T_S with n leaves.

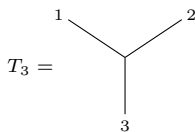


Fig. 3. Labeled trivalent tree with three leaves.

Algorithm 1: Associating a trivalent tree T_S to an iterated sequence S .

Input: An iterated sequence $S = (\varepsilon_{i_n} - \varepsilon_n, \varepsilon_{j_n} - \varepsilon_n, \dots, \varepsilon_{i_3} - \varepsilon_3, \varepsilon_{j_3} - \varepsilon_3)$, the trivalent tree T_3 as in Fig. 3.

Initialization: Set $k = 4, T_3^S := T_3$.

for $k \leq n$ **do**

Construct a tree T_k^S from T_{k-1}^S by replacing the edge with leaf i_k in T_{k-1}^S by three edges forming a cherry with leaves labeled by i_k and k .

if $k = n$ **then**

Output: The tree T_n^S .

else

Replace k by $k + 1, T_{k-1}^S$ by T_k^S and start over.

Output: The tree $T_S := T_n^S$ and the sequence $\mathbb{T}_S := (T_n^S, \dots, T_3^S)$ of trees.

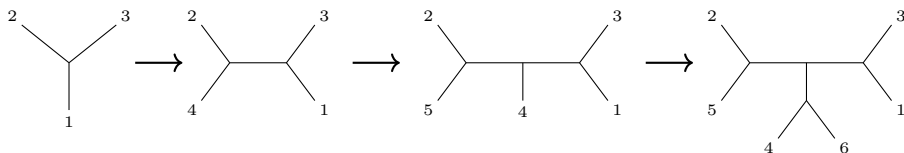


Fig. 4. The sequence \mathbb{T}_S for S as in Example 4.

Definition 13. To an iterated sequence S we associate the trivalent tree T_S and the sequence of trees $\mathbb{T}_S = (T_n^S, \dots, T_3^S)$ that are the output of Algorithm 1. Denote by C_S the maximal cone in $\text{trop}(\text{Gr}(2, \mathbb{C}^n))$ corresponding to the tree T_S by [21, Theorem 3.4].

Example 4. Consider $S = (\varepsilon_4 - \varepsilon_6, \varepsilon_5 - \varepsilon_6, \varepsilon_2 - \varepsilon_5, \varepsilon_3 - \varepsilon_5, \varepsilon_2 - \varepsilon_4, \varepsilon_3 - \varepsilon_4, \varepsilon_1 - \varepsilon_3, \varepsilon_2 - \varepsilon_3)$, an iterated sequence for $\text{Gr}(2, \mathbb{C}^6)$. We construct the trees $\mathbb{T}_S = (T_3^S, T_4^S, T_5^S, T_6^S)$ by Algorithm 1. Fig. 4 shows the obtained sequence of trees.

Let $M_S := M_{\mathbf{v}_S}$ be the weighting matrix associated to \mathbf{v}_S as in Definition 5. We compute the initial ideal $\text{in}_{M_S}(\mathcal{I}_{2,n})$ of the Plücker ideal $\mathcal{I}_{2,n}$ to apply [8, Theorem 1].

Proposition 1. For every iterated sequence S we have $\text{in}_{M_S}(\mathcal{I}_{2,n}) = \text{in}_{C_S}(\mathcal{I}_{2,n})$.

Proof. Let $\{e_{ij}\}_{(i < j) \in \mathcal{I}_{2,n}}$ be the standard basis of $\mathbb{R}^{\binom{n}{2}}$. Adopting the notation for monomials in the polynomial ring $\mathbb{C}[p_{ij}]_{i < j}$ we have

$$R_{i,j,k,l} = p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk} = \mathbf{p}^{e_{ij}+e_{kl}} - \mathbf{p}^{e_{ik}+e_{jl}} + \mathbf{p}^{e_{il}+e_{jk}}.$$

In particular, $M_S(e_{ij} + e_{kl}) = \mathbf{v}_S(\bar{p}_{ij}) + \mathbf{v}_S(\bar{p}_{kl}) = \mathbf{v}_S(\bar{p}_{ij}\bar{p}_{kl})$ and $\text{in}_{M_S}(R_{i,j,k,l})$ is the sum of those monomials in $R_{i,j,k,l}$ for which the valuation \mathbf{v}_S of the corresponding monomials in $A_{2,n}$ is minimal with respect to \prec_Ψ .

Recall that $\text{in}_{C_S}(\mathcal{I}_{2,n}) = \langle \text{in}_{C_S}(R_{i,j,k,l}) \mid 1 \leq i < j < k < l \leq n \rangle$ by [21, Proof of Theorem 3.4]. This implies that it is enough to prove the following claim.

Claim. For every Plücker relation $R_{i,j,k,l}$ with $1 \leq i < j < k < l \leq n$ we have $\text{in}_{C_S}(R_{i,j,k,l}) = \text{in}_{M_S}(R_{i,j,k,l})$.

Proof of claim. We proceed by induction. For $n = 4$ let $S = (\varepsilon_i - \varepsilon_4, \varepsilon_j - \varepsilon_4, \varepsilon_{i_3} - \varepsilon_3, \varepsilon_{j_3} - \varepsilon_3)$, i.e. the tree T_S has a cherry labeled by i and 4 . Consider the Plücker relation $R_{i,j,k,4} = p_{ij}p_{k4} - p_{ik}p_{j4} + p_{i4}p_{jk}$ with $\{i, j, k\} = [3]$. Then

$$\text{in}_{C_S}(R_{i,j,k,4}) = p_{ij}p_{k4} - p_{ik}p_{j4}.$$

Let $S' = (\varepsilon_{i_3} - \varepsilon_3, \varepsilon_{j_3} - \varepsilon_3)$ be the sequence for $\text{Gr}(2, \mathbb{C}^3)$ and denote by \hat{p}_{rs} with $(r < s) \in I_{2,3}$ the Plücker coordinates in $A_{2,3}$. For $m = (m_{d-2}, \dots, m_1) \in \mathbb{Z}^{d-2}$ and $m_d, m_{d-1} \in \mathbb{Z}$ write $(m_d, m_{d-1}, m) := (m_d, m_{d-1}, m_{d-2}, \dots, m_1)$. We compute

$$\mathbf{v}_S(\bar{p}_{i4}) = (0, 1, \mathbf{v}_{S'}(\hat{p}_{ij})), \quad \mathbf{v}_S(\bar{p}_{j4}) = (1, 0, \mathbf{v}_{S'}(\hat{p}_{ij})), \quad \text{and} \quad \mathbf{v}_S(\bar{p}_{k4}) = (1, 0, \mathbf{v}_{S'}(\hat{p}_{ik})).$$

This implies $\mathbf{v}_S(\bar{p}_{i4}\bar{p}_{jk}) \succ_\Psi \mathbf{v}_S(\bar{p}_{ij}\bar{p}_{k4}) = \mathbf{v}_S(\bar{p}_{ik}\bar{p}_{j4})$, and hence $\text{in}_{M_S}(R_{i,j,k,4}) = \text{in}_{C_S}(R_{i,j,k,4})$.

Assume the claim is true for $n - 1$ and let $S = (\varepsilon_{i_n} - \varepsilon_n, \varepsilon_{j_n} - \varepsilon_n, \dots, \varepsilon_{i_3} - \varepsilon_3, \varepsilon_{j_3} - \varepsilon_3)$ be an iterated sequence for $\text{Gr}(2, \mathbb{C}^n)$. Then $S' = (\varepsilon_{i_{n-1}} - \varepsilon_{n-1}, \varepsilon_{j_{n-1}} - \varepsilon_{n-1}, \dots, \varepsilon_{i_3} - \varepsilon_3, \varepsilon_{j_3} - \varepsilon_3)$ is an iterated sequence for $\text{Gr}(2, \mathbb{C}^{n-1})$. Denote by \hat{p}_{ij} with $(i < j) \in I_{2,n-1}$ the Plücker coordinates in $A_{2,n-1}$. As $\mathbf{v}_S(\bar{p}_{ij}) = (0, 0, \mathbf{v}_{S'}(\hat{p}_{ij}))$ for $i, j < n$ we deduce $\text{in}_{C_S}(R_{i,j,k,l}) = \text{in}_{M_S}(R_{i,j,k,l})$ with $i, j, k, l < n$ by induction. Consider a Plücker relation of form $R_{i,j,k,n}$ and compute

$$\mathbf{v}_S(\bar{p}_{ln}) = \begin{cases} (1, 0, \mathbf{v}_{S'}(\hat{p}_{ri_n})), & \text{if } l \neq i_n, \\ (0, 1, \mathbf{v}_{S'}(\hat{p}_{i_n j_n})), & \text{if } l = i_n. \end{cases}$$

As (i_n, n) is a cherry in T_S we observe that the associated weight vector $w_{T_S} \in C_S^\circ \subset \text{trop}(\text{Gr}(2, \mathbb{C}^n))$ satisfies $(w_{T_S})_{ln} = (w_{T_S})_{li_n} = (w_{T_{S'}})_{li_n} - 1$. In particular, for $i, j, k \neq i_n$ we have by induction $\text{in}_{M_S}(R_{i,j,k,n}) = \text{in}_{C_S}(R_{i,j,k,n})$. The only relations left to consider are of form $R_{i_n,j,k,n}$ for $j, k \in [n - 1] \setminus \{i_n\}$. For M_S we compute by the above

$$\mathbf{v}_S(\bar{p}_{i_n j} \bar{p}_{kn}) = \mathbf{v}_S(\bar{p}_{i_n k} \bar{p}_{jn}) \succ_\Psi \mathbf{v}_S(\bar{p}_{i_n n} \bar{p}_{jk}).$$

Hence, $\text{in}_{M_S}(R_{i_n,j,k,n}) = p_{i_n j} p_{kn} - p_{i_n k} p_{jn}$. As (i_n, n) labels a cherry in T_S we also have $\text{in}_{C_S}(R_{i_n,j,k,n}) = \text{in}_{M_S}(R_{i_n,j,k,n})$. \square

Theorem 2. For every iterated sequence S we have $\text{gr}_S(A_{2,n}) \cong \mathbb{C}[p_{ij}]_{i < j} / \text{in}_{C_S}(\mathcal{I}_{2,n})$. Conversely, for every maximal prime cone $C \subset \text{trop}(\text{Gr}(2, \mathbb{C}^n))$ there exists a birational sequence S , such that $\mathbb{C}[p_{ij}]_{i < j} / \text{in}_C(\mathcal{I}_{2,n}) \cong \text{gr}_S(A_{2,n})$.

Proof. Let S be an iterated sequence. By Corollary 1 we can apply [8, Theorem 1] and get

$$\text{gr}_S(A_{2,n}) \cong \mathbb{C}[p_{ij}]_{i < j} / \text{in}_{M_S}(\mathcal{I}_{2,n}) \stackrel{\text{Proposition 1}}{=} \mathbb{C}[p_{ij}]_{i < j} / \text{in}_{C_S}(\mathcal{I}_{2,n}).$$

For the second we pursue by induction: for $n = 3$ there is only one maximal cone $C \subset \text{trop}(\text{Gr}(2, \mathbb{C}^3))$. The sequence $S = (\varepsilon_1 - \varepsilon_3, \varepsilon_2 - \varepsilon_3)$ satisfies $C_S = C$. Assume the claim is true for $n - 1$ and consider a maximal cone $C \subset \text{trop}(\text{Gr}(2, \mathbb{C}^n))$ with associated tree T_C . Let \mathbf{T}_C denote the underlying unlabeled tree. Find a cherry in \mathbf{T}_C and remove it. Denote by \mathbf{T}'_C the obtained tree with $n - 1$ leaves. By induction there exist an iterated sequence S' for $\text{Gr}(2, \mathbb{C}^{n-1})$ with associated tree $T_{S'}$ of shape \mathbf{T}'_C . Now add back the cherry and label the additional leaf by n . The cherry is labeled by some $i < n$ and n . Hence, $S := (\varepsilon_i - \varepsilon_n, \varepsilon_j - \varepsilon_n, S')$ for arbitrary $j < n$ with $i \neq j$ is an iterated sequence with T_S of shape \mathbf{T}_C and therefore $\text{gr}_S(A_{2,n}) \cong \mathbb{C}[p_{ij}]_{i < j} / \text{in}_C(\mathcal{I}_{2,n})$. \square

For an iterated sequence S for $\text{Gr}(2, \mathbb{C}^n)$ denote by \mathbf{T}_i^S the (unlabeled) trivalent tree underlying the labeled trivalent tree T_i^S with i leaves in the tree sequence \mathbb{T}_S . Algorithm 1 provides a tool for comparing whether two iterated sequences induce isomorphic flat toric degenerations. Construct $\mathbb{T}_{S_1}, \mathbb{T}_{S_2}$ for two such sequences S_1, S_2 and consider $\mathbf{T}_n^{S_1}$ and $\mathbf{T}_n^{S_2}$. If $\mathbf{T}_n^{S_1}$ and $\mathbf{T}_n^{S_2}$ coincide then

$$\text{in}_{T_n^{S_1}}(\mathcal{I}_{2,n}) \cong \text{in}_{T_n^{S_2}}(\mathcal{I}_{2,n}). \tag{4.2}$$

Let $\mathcal{H}_d := \text{conv}(e \in \{0, 1\}^d)$ be the hypercube in $\mathbb{R}_{\geq 0}^d$.

Corollary 2. For every iterated sequence S for $\text{Gr}(2, \mathbb{C}^n)$ the Newton-Okounkov polytope has $\binom{n}{2}$ vertices of form $\mathbf{v}_S(\bar{p}_{ij})$ for $1 \leq i < j \leq n$. Further, it has no interior lattice points and satisfies

$$\Delta(A_{2,n}, \mathbf{v}_S) \subset \mathcal{H}_d.$$

Moreover, if S_1, S_2 are two different iterated sequences for $\text{Gr}(2, \mathbb{C}^n)$ with $\mathbf{T}_n^{S_1} = \mathbf{T}_n^{S_2}$, then the Newton-Okounkov polytopes are unimodularly equivalent.⁵

Proof. By Theorem 2 the value semi-group $S(A_{2,n}, \mathbf{v}_S)$ is generated by $\{\mathbf{v}_S(\bar{p}_{ij})\}_{i < j}$. This implies $\Delta(A_{2,n}, \mathbf{v}_S) = \text{conv}(\mathbf{v}(\bar{p}_{ij}))_{i < j}$. Observe from the proof of Proposition 1

⁵ Two polytopes $P, Q \subset \mathbb{R}^d$ are called *unimodularly equivalent* if there exists matrix $M \in GL_d(\mathbb{Z})$ and $w \in \mathbb{Z}^d$ $Q = f_M(P) + w$, where $f_M(x) = xM$ for $x \in \mathbb{R}^d$. It is denoted $Q \cong P$ and by [10, §2.1 and §2.3] implies that the associated projective toric varieties are isomorphic.

that $\mathbf{v}_S(\bar{p}_{ij}) \in \{0, 1\}^d$. Therefore, $\mathbf{v}_S(\bar{p}_{ij}) \notin \text{conv}(\mathbf{v}_S(\bar{p}_{kl}) | (k < l) \neq (i < j))$. Hence, for all $1 \leq i < j \leq n$ every $\mathbf{v}_S(\bar{p}_{ij})$ is indeed a vertex and $\Delta(A_{2,n}, \mathbf{v}_S) \subset \mathcal{H}_d$.

Regarding the second statement, let S_1, S_2 be two iterated sequences. Assume we have $T_n^{S_1} = T_n^{S_2}$ and consider the lattice ideals associated to each in $C_{S_i}(\mathcal{I}_{2,n})$ (see e.g. [6, Lemma 2]). The corresponding lattice points $\{l_j^i\}_{j=1, \dots, \binom{n}{2}}$ define polytopes in their ambient lattice which are unimodularly equivalent to $\Delta(A_{2,n}, \mathbf{v}_{S_i})$ (see [6, proof of Theorem 4] for the precise construction). Therefore, (4.2) implies $\Delta(A_{2,n}, \mathbf{v}_{S_1}) \cong \Delta(A_{2,n}, \mathbf{v}_{S_2})$. \square

Given a maximal cone $C_T \in \text{trop}(\text{Gr}(2, \mathbb{C}^n))$ with associated labeled trivalent tree T the rays of C are in bijection with *internal edges* of T , i.e. those not adjacent to any leaf. Internal edges give rise to partitions of $[n]$ into two sets (each of cardinality ≥ 2) corresponding to the labelings of the leaves on either side of the edge. Given a partition (A, B) of $[n]$ the ray is $E_{A,B} = -\sum_{i \in A} \sum_{j \in B} e_{ij} \in \mathbb{R}^{\binom{n}{2}}$, see [21, page 396].

Corollary 3. *Let S be an iterated sequence S for $\text{Gr}(2, \mathbb{C}^n)$ and for $n \geq 4$ we consider the projection $\pi_n : \mathbb{R}^{\binom{n}{2}} \rightarrow \mathbb{R}^{\binom{n-1}{2}}$ onto all coordinates that do not involve n . Then $\mathbb{T}_S = (T_3, T_4^S, \dots, T_n^S)$ corresponds to a sequence of cones $(C_{T_3}, C_{T_4^S}, \dots, C_{T_n^S})$ with $C_{T_i^S} \in \text{trop}(\text{Gr}(2, \mathbb{C}^i))$ and $\pi_i(C_{T_i^S}) = C_{T_{i-1}^S}$.*

Proof. The first property follows from Theorem 2. For the second notice that if (A, B) is a partition of $[n]$ then $(A \cap [n-1], B \cap [n-1])$ is a partition of $[n-1]$. Moreover, $\pi_n(E_{A,B}) = E_{A \cap [n-1], B \cap [n-1]} \in \mathbb{R}^{\binom{n-1}{2}}$. For every $4 \leq i \leq n$ the trees T_i^S and T_{i-1}^S have all interior edges in common, except the one adjacent to the cherry added when constructing T_i^S from T_{i-1}^S . Hence, the rays of $C_{T_i^S}$ are mapped to the rays of $C_{T_{i-1}^S}$ by π_n . \square

Remark 3. In [16, Proposition 8.16] the authors describe an initial ideal $\text{in}_M(I)$ for an ideal $I \subset \mathbb{C}[x_1, \dots, x_n]$ with respect to a matrix $M \in \mathbb{R}^{n \times d}$ as $\text{in}_{u_1}(\dots \text{in}_{u_d}(I) \dots)$, where u_1, \dots, u_d are the columns of M . This leads to a sequence of initial ideals of I , namely $I, \text{in}_{u_d}(I), \text{in}_{u_{d-1}}(\text{in}_{u_d}(I))$ and so on. The sequence of cones $(C_{T_3}, C_{T_4^S}, \dots, C_{T_n^S})$ from Corollary 3 also yields a sequence of initial ideals $\text{in}_{T_3}(\mathcal{I}_{2,3}), \text{in}_{T_4^S}(\mathcal{I}_{2,4}), \dots, \text{in}_{T_n^S}(\mathcal{I}_{2,n})$. This sequence is different in nature from the first as we are taking initial ideals of different ideals.

The following definition allows us to interpret iterated sequences for $\text{Gr}(2, \mathbb{C}^n)$ in a combinatorial way in Corollary 4 below.

Definition 14. The *tree graph* \mathcal{T} is an infinite graph whose vertices at level $i \geq 3$ correspond to trivalent trees with i leaves. There is an arrow $T \rightarrow T'$, if T has i leaves, T' has $i + 1$ leaves and T' can be obtained from T by attaching a new boundary edge to the middle of some edge of T . There is a unique source T_3 at level 3 (see Fig. 1).

Corollary 4. *Every iterated sequence S for $\text{Gr}(2, \mathbb{C}^n)$ corresponds to a path from T_3 to T_n^S in the tree graph \mathcal{T} .*

Proof. The underlying unlabeled trees in the sequence $\mathbb{T}_S = (T_3, T_4^S, \dots, T_n^S)$ associated to S define the path $T_3 \rightarrow T_4^S \rightarrow \dots \rightarrow T_n^S$ in \mathcal{T} . \square

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