Differential Algebra Lectures

François Boulier
University of Lille, CRIStAL, France

Lecture notes are available at

https://codeberg.org/francois.boulier/DifferentialAlgebra/src/branch/main/gallery/resources/An-Introduction-to-Differential-Algebra.pdf

The whole set of lectures is dedicated to an algorithmic elimination theory in Ritt and Kolchin differential algebra which actually provides simplification tools for systems of polynomial ODE and PDE, possibly depending on parameters.

Lecture 1 - Applications of Differential Algebra

The first lecture shows some problems where these tools are relevant and illustrates them by means of computer algebra software:
1. parameter estimation: eliminate variables for which measures are not available, symbolic integration of the resulting input-output equation.
2. model reduction: in the case of fast/slow dynamics, it is useful to consider a limit system obtained by annihilating a parameter in a parametric initial system. The limit system then needs some simplification.

The last example will show some relationship between differential elimination, numerical integration and the existence and uniqueness of formal power series solutions, which is directly related to the main topic of the school.

All these examples permit us to introduce informally some differential algebra terminology such as characteristic sets and regular differential chains.

Lecture 2 - Formal Power Series and Differential Elimination

The relationship is subtle. Differential elimination, applied to an input differential polynomial system S, solves the following problem: does there exist initial values leading to formal power series solutions of S? In this context, characteristic sets and regular chains can be viewed as systems simplified enough to expand formal power series solutions in a straightforward manner. In tropical differential geometry, the addressed underlying problem would rather be: given S and initial values, does there exist formal power series solutions of S for these initial values? Some of these problems are algorithmically decidable, some other ones are not and ... one case is not clear. An important related paper is Denef and Lipshitz (1984) one.

Lecture 3 - An Equivalence Theorem for Regular Chains

Differential characteristic sets and regular differential chains are particular cases of characteristic sets and regular chains, in commutative algebra. The two notions (characteristic sets and regular chains) are distinct and do not address exactly the same purposes. Characteristic sets were introduced by Ritt in 1950. Regular chains are more recent (1990) and their theory is slightly more complicated. However, for the elimination theory of polynomial systems and for practical problems, regular chains prove to be the right generalization of characteristic sets. The third lecture focuses on an equivalence theorem for regular chains. A simplified proof was obtained while preparing the lecture notes. It is given in section 3.11.

Lecture 4 - An Elimination Theory in Differential Algebra

In this last lecture, the equivalence theorem for regular chains is generalized to the context of differential polynomial systems. This generalization requires the introduction of squarefree regular chains, Lazard's Lemma, Ritt's reduction algorithm and, in the partial differential (PDE) case, Rosenfeld's Lemma (1959). The principle of an elimination procedure such as the Rosenfeld-Groebner algorithm is sketched.

Tropical Geometry Lectures

Lucía López de Medrano
Instituto de Matemáticas Unidad Cuernavaca
UNAM

The goal of this course is to present the basic concepts of tropical geometry.

Although tropical geometry has its beginnings in the year 2000, tropical algebra, or max-plus algebra, on which it is based, dates from the 1970s. In this course we will see the background, the so-called fundamental theorems of tropical algebra and tropical geometry and the basic definitions necessary to understand the courses of the second week of this school.

Computational Commutative Algebra Lectures

Matías Bender
INRIA Saclay – Île-de-France - CMAP,
École polytechnique in Palaiseau, France

This course offers an introduction to fundamental concepts in algebra and geometry with a focus on computational methods. Our primary tools will be Groebner bases and resultants. The objectives of the course are twofold: firstly, to employ these tools to study the properties of ideals and varieties. Secondly, to develop effective algorithms for manipulating these objects. Additionally, the course includes applications of these techniques, with a specific emphasis on elimination theory.

Tropical Differential Algebra Lectures

Cristhian Garay Lopez and Carla Valencia Negrete
CONACYT Research fellow CIMAT Guanajuato // Universidad Iberoamericana, Ciudad de México

In this course we pretend to motivate the usefulness of algebraic and tropical methods to find some kinds of solutions (formal power series solutions) of systems of algebraic differential equations.

It consists of two parts. In the first one we will give an initial motivation in the form of a concrete example of modelation of complex systems through systems of linear ordinary differential equations, of which we will analyze the set of formal power series solutions. We will also show using this example that the tropical differential algebraic framework distinguishes in a novel way between complex systems with periodical solutions and others that do not have them.

In the second part, we will discuss a generalization of this setting by introducing some aspects of the theory of tropical differential algebra. We will see that the set of all formal power series solutions associated with systems of algebraic differential equations have a strong combinatorial structure which can be modeled algebraically using tropical mathematics, and this point of view gives new and interesting insights about these sets of solutions.

Topics in Differential Algebra Lectures

Sebastian Falkensteiner
Max Planck Institute for Mathematics in the Sciences
Leipzig, Germany

Representing and computing solutions of algebraic differential equations

Systems of differential equations are notoriously hard to solve. Many systems of differential equations do not admit closed form solutions in elementary functions and hence cannot be solved symbolically. Despite this, we study some approaches where, under as mild additional assumptions as possible, symbolic solutions can be derived or at least a better implicit representation of them can be given. Additionally, there are increasingly good heuristics implemented in computer algebra systems to find solutions. Given such a set of heuristically found solutions, the question remains whether this set is the complete solution set. We will study this by using the notions of the size of solution sets.

Newton’s Method for Solving Differential Equations Lectures

Fuensanta Aroca
Instituto de Matemáticas
UNAM

Talk: CTLN's from a tropical viewpoint

Vinicio Gómez
Facultad de Ciencias
UNAM

This talk is about a certain family of Ordinary Differential Equations Systems, named Combinatorial Threshold Linear Networks (CTLN’s for short). On one hand, these systems are related with Graph Theory, Combinatorics and Hyperplane Arrangements. On the other hand, they can reproduce many nonlinear phenomena: many stable equilibria points, limit cycles, and chaos. Can tropical methods be applied to these systems? We will try to give a first discussion about this.

Talk: The Fundamental theorem of tropical differential algebra over nontrivially valued fields

Stefano Mereta
Max Planck Institute for Mathematics in the Sciences
Leipzig, Germany

We will discuss a fundamental theorem for tropical differential equations analogue of the fundamental theorem of tropical geometry in this context. We extend results from Aroca et al. and from Fink and Toghani, holding only in the case of trivial valuation as introduced by Grigoriev, to differential equations with power series coefficients over any valued field. To do so, a crucial ingredient is the framework for tropical differential equations introduced by Giansiracusa and Mereta. As a corollary of the fundamental theorem, the radius of convergence of solutions of a differential equation over a nontrivially valued field can be computed tropically. We will also discuss an explicit example. This talk is based on results appearing in arXiv:2303.12124.

Talk: Tropical linear spaces: finite and infinite

Alejandro Vargas
Laboratorio de Matemáticas Jean Leray
University de Nantes, France

We introduce the theory of tropical linear spaces in the finite setting for the non-specialist audience. We begin by asking how a linear subspace of a vector space can be encoded. The two usual ways are as the span of a finite set of vectors, or as the solution set of a family of equations. Both involve making a choice or remembering more information than needed. We fix this by considering the plücker embedding, or the linear forms of minimal support, respectively. Both roads lead us, in a dual manner, to the combinatorial concept of a valuated matroid. This fundamental object encodes all the information of a tropical linear space. We describe its initial matroids, lattice of flats, and the associated order complex that locally describes the polyhedral structure of the tropical linear space. After this introduction, we move to the infinite setting. We are motivated by the theory of tropical differential equations, whose power series solutions live in a vector space of infinite dimension. We explore how things break down in this setting, and report on ongoing work with Garay and Mereta about potential fixes and first steps towards a theory of infinite valuated matroids.

Talk: A little bit of real tropical geometry

Benoît Bertrand
Institut de mathématiques de Toulouse
Université Paul Sabatier Toulouse III, France

In part, tropical geometry can be understood as a limit of complex varieties. Mikhalkin's theorem gives a correspondence between planar complex curves and tropical curves. Simultaneously Mikhalkin gave a correspondence between real algebraic curves and tropical curves decorated by signs which we call real tropical curves.

I will try to explain properties of real tropical curves with pictures.

Talk: Deducting Conserved Quantities for Numerical Schemes using Parametric Gröbner Systems

Bashayer Majrashi
King Abdullah University of Science and Technology
Thuwal, Saudi Arabia

In partial differential equations (PDEs), conserved quantities like mass and momentum are fundamental to understanding the behavior of the described physical systems.
The preservation of conserved quantities is essential when using numerical schemes to approximate solutions of corresponding PDEs. If the discrete solutions obtained through these schemes fail to preserve the conserved quantities, they may be physically meaningless and unreliable.

Previous approaches focused on checking conservation in PDEs and numerical schemes, but they did not give adequate attention to systematically handling parameters. This is a crucial aspect because many PDEs and numerical schemes have parameters that need to be dealt with systematically.
Here, we investigate if the discrete analog of a conserved quantity is preserved under the solution induced by a parametric finite difference method.

In this talk, we introduce an algorithm to effectively and reliably deduce conserved quantities in the context of parametric schemes, using the concept of comprehensive Gröbner systems.

The main contribution of this work is the development of a versatile algorithm capable of handling various parametric explicit and implicit schemes, higher-order derivatives, and multiple spatial dimensions. The algorithm's effectiveness and efficiency are demonstrated through examples and applications.
In particular, we illustrate the process of selecting an appropriate numerical scheme among a family of potential discretization for a given PDE.