TROPICAL GEOMETRY EXERCISES SHEET.

1. Semirings and idempotent semirings

A **semiring** (commutative with unit) is a 5-tuple $S = (S, \times, +, 1, 0)$ which is a ring, except that $(S, +, 0)$ is just a monoid, and not necessarily a group.
We also demand that $0 \times a = a \times 0 = 0$ for every $a \in S$.

A semiring S is **idempotent** if $a + a = a$ for all $a \in S$.

(1) Show that the following are idempotent semirings.
  (a) $S = \mathcal{P}(X)$ (the power set of a set $X \neq \emptyset$) together with union and intersection.
  (b) $S = \mathcal{P}_m := \{\mathrm{Conv}(A) \,:\, A \subset \mathbb{N}^m \text{ finite}\}$. We define
   - $\mathrm{Conv}(S_1) \oplus \mathrm{Conv}(S_2) := \mathrm{Conv}(S_1 \cup S_2)$, and
   - $\mathrm{Conv}(S_1) \odot \mathrm{Conv}(S_2) := \mathrm{Conv}(S_1) + \mathrm{Conv}(S_2)$ (this is Minkowski sum).

  (c) $S = \mathbb{N}^{gcd} := (\mathbb{N}, \times, +_{gcd}, 1, 0)$, where $\times$ is the usual product and $a +_{gcd} b := gcd(a, b)$

(2) Let S be an idempotent semiring. Define a binary relation $\mathfrak{R} \subset S \times S$ by $a \mathfrak{R} b$ if and only if $a + b = b$.
  (a) Show that $\mathfrak{R}$ is an order relation on S, so we will denote it by $\leq_S$, or just by $\leq$.
  (b) Show that the unit ball $S_{\leq 1} := \{x \in S \,:\, x \leq 1\}$ is a semiring.
  (c) Describe explicitly the order $\leq_S$ and the unit ball $S_{\leq 1}$ for each one of the idempotent semirings S from Exercise (1).

(3) Let $\mathbb{T} = (\mathbb{R} \cup \{-\infty\}, \times, +)$ be the tropical semiring.
  (a) Describe the order $\leq_{\mathbb{T}}$ and the unit ball $\mathbb{T}_{\leq 1}$.
  (b) The same for $S = \mathbb{T}[x_1, \ldots, x_n]$.

2. Commutative algebra for semirings

Let S be a semiring

(4) An equivalence relation $\mathfrak{R} \subset S \times S$ is a **semiring congruence** if $a \mathfrak{R} b$ implies $(a + c) \mathfrak{R} (b + c)$ and $(ac) \mathfrak{R} (bc)$ for every $c \in S$.
  (a) Show that the set of equivalence classes $S/\mathfrak{R}$ is a semiring, and that the projection map $\pi : S \to S/\mathfrak{R}$ is a homomorphism of semirings,
  (b) Show that $\sim \subset \mathbb{T}[x_1, \ldots, x_n] \times \mathbb{T}[x_1, \ldots, x_n]$ defined by $f \sim g$ iff $ev_f = ev_g$ is a semiring congruence.
  (*c) The semiring $\mathbb{T}[x_1, \ldots, x_n]/\sim$ is the semiring of tropical polynomial functions on $\mathbb{T}^n$. Describe $\mathbb{T}[x]/\sim$ explicitly.

(5) We say that $I \subset S$ is an **ideal** if it is closed under addition and it is absorbent with respect to product.
  (a) Show that if $\mathfrak{R} \subset S \times S$ is a semiring congruence, then

$$\mathbb{I}(\mathfrak{R}) := \{x \in R \,:\, (x, 0) \in \mathfrak{R}\}$$

  is an ideal.

(b) Describe $\mathbb{I}(\sim) \subset \mathbb{T}[x_1, \dots, x_n]$ from Exercise (4b).

(6) An ideal $\mathfrak{p} \subset S$ is **prime** if $S \setminus \mathfrak{p}$ is closed under product. On $S \times (S \setminus \mathfrak{p})$, define the binary relation

$$(a, b) \sim (c, d) \Leftrightarrow \text{ there exists } t \in S \setminus \mathfrak{p} \text{ such that } tad = tbc. \tag{1}$$

(a) Show that (1) is an equivalence relation.

We denote by $S_{\mathfrak{p}} := [S \times (S \setminus \mathfrak{p})]/\sim$ the set of equivalence classes, and by $\frac{a}{b}$ the equivalence class of $(a, b) \in S \times (S \setminus \mathfrak{p})$ in $S_{\mathfrak{p}}$.

(b) Show that $S_{\mathfrak{p}}$ endowed with the usual operations of fractions, is a semiring.

(*c) Show that $(0) \subset \mathbb{T}[x_1, \dots, x_n]$ is a prime ideal, and describe the elements of $\mathbb{T}(x_1, \dots, x_n) := \mathbb{T}[x_1, \dots, x_n]_{(0)}$ as rational functions on $\mathbb{T}^n$.

(7) Is the ideal $\langle x^2 + x^3, x^4 \rangle \subset \mathbb{T}[x]$ principal?

3. Non-Archimedean seminorms

A **non-Archimedean seminorm** is a triple (R, v, S) consisting of a ring R, an idempotent semiring S (see Section 1), and a function $v : R \to S$. This triple must satisfy the following conditions for all $a, b \in R$:

- $v(\pm 1) = 1$, $v(0) = 0$,
- $v(ab) \leq v(a)v(b)$,
- $v(a + b) \leq v(a) + v(b)$

(8) Consider the function

$$New : K[x_1, \dots, x_n] \mapsto \mathcal{P}_n$$

sending a polynomial $f = \sum_I a_I x^I$ to its Newton polytope $New(f) = Conv(Supp(f))$. Show that it is a non-Archimedean seminorm. See Exercises (1b) and (2a).

4. Tropical geometry and tropicalization

(9) Let $h, g \in \mathbb{T}[x, y]$ with $h \neq g$.

(a) is it true that $V(h) \cup V(g) = V(hg)$?

(b) Show that $V(h) \cap V(g)$ needs not to be a finite set, nor a tropical curve,

(c) Let $\langle h, g \rangle \subset \mathbb{T}[x, y]$ be the ideal generated by $\{h, g\}$. Is

$$\bigcap_{f \in \langle h, g \rangle} V(f)$$

always a finite number of points?

(10) Let K be a field and let $\Gamma = (\Gamma, \times, 1, \leq)$ be a totally ordered abelian group contained in $\mathbb{R}$. Define

$$K[t^\Gamma] = \left\{ \sum_{i \in S} \alpha_i t^i \ : \ \alpha_i \in K, \ S \subset \Gamma \text{ well-ordered} \right\}.$$

(a) Show that $K[t^\Gamma]$ is a field (endowed with the usual operations of addition and convolution product).

(b) Show that $ord_t : K[t^\Gamma] \to \Gamma$ is a non-Archimedean seminorm.

(c) Consider the function

$$trop : K[x_1, \ldots, x_n] \mapsto \mathbb{T}[x_1, \ldots, x_n]$$

sending a polynomial $f = \sum_I a_I x^I$ to the tropical polynomial $trop(f) = \sum_I ord_t(a_I)x^I$. Show that it is a non-Archimedean seminorm. See Exercise (8).

Let $I \subset K[x_1, \ldots, x_n]$ be an ideal. Its **tropicalization** is $trop(I) = \langle trop(f) : f \in I \rangle$.

(11) Let $I = \langle x + y, x - y \rangle \subset K[x, y]$. Show that $f = x^2 - y^2 \in trop(I)$, but $f \notin \langle trop(x + y), trop(x - y) \rangle$.