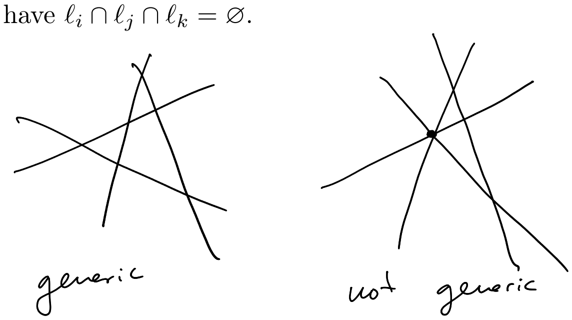
Exercises on tropicalized linear spaces and tropical linear spaces.

For $n \in \mathbb{Z}_{\geq 1}$, denote by $[n]$ the set $\{1, \dots, n\}$.
Our running example in the talk is W, the plane in $\mathbb{R}^5$ passing through the points:

$$a = (1, 1, 0, 2, 1)$$
$$b = (0, 2, 1, 0, 4)$$
$$c = (0, 1, 1, 0, 3)$$

(1) A dimension d affine space is defined by $d + 1$ points in general position. Given an affine space defined by points, we want to test if a given point lies in it.
  (a) Is $(1, 2, 3, 4)$ in the line defined by $(5, 6, 7, 8)$ and $(9, 10, 11, 12)$?
  (b) Calculate a parameter α such that the point $(2, 4, 6, 8, 1)$ is in the line defined by $(5, 6, 7, 8, \alpha)$ and $(9, 10, 11, 12, \alpha)$ in $\mathbb{R}^5$.
  (c) Calculate a parameter β such that the point $(2, 4, 6, 8, 1)$ is in the line defined by $(1, 1, 1, 1, 0)$ and $(1, 1, 1, 1, \beta)$ in $\mathbb{R}^5$.

(2) Let $\mathcal{L} = \{\ell_1, \dots, \ell_n\}$ be a family of n lines in $\mathbb{R}^2$, which are in *generic* position, that is for every three distinct indices i, j, k we have $\ell_i \cap \ell_j \cap \ell_k = \varnothing$.



Let X be the complement of the lines, that is the following set:

$$X = \mathbb{R}^2 \setminus \bigcup_{i \in [n]} \ell_i.$$

For n equals 3 the number of connected components of X is 7.
  (a) Calculate by making examples the number of connected components for $n = 4, 5, 6$.
  (b) Conjecture a formula. Try to prove it (*).
  (c) Calculate the number of connected components of $W \cap (\mathbb{R} \setminus \{0\})^5$, compare with the previous answer. (*)
  (d) Is X an algebraic variety? (*)

(3) Let W' be the rowspan of

$$\begin{pmatrix} 2 & 1 & 1 & 1 & 1 \\ 4 & 2 & 3 & 2 & 1 \end{pmatrix}$$

Calculate generators for the ideal $I(W')$.

(4) Let $\Sigma \subset \mathbb{R}^4$ be the tropical variety of points $p = (p_1, p_2, p_3, p_4)$ such that for any triple of distinct indices i, j, k the minimum is achieved twice in

$$\min(p_i, p_j, p_k).$$

Let $\tilde{\Sigma} = \Sigma \cap \{x_1 + x_2 + x_3 + x_4 = 0\}$.
  (a) Calculate the dimension of $\tilde{\Sigma}$.
  (b) Calculate the number of 1-cells of $\tilde{\Sigma}$.
  (c) Add together the primitive vectors of the 1-cells of $\tilde{\Sigma}$ and verify the balancing condition.
  (d) Generalize from $\mathbb{R}^4$ to $\mathbb{R}^n$.

(5) Let $I = \langle 2x_4 - x_1, x_3 - 2x_4 + x_5, x_2 - x_4 \rangle$ be an ideal in $\mathbb{C}[x_1, x_2, x_3, x_4]$.
  (a) Find parameters $\alpha, \beta \in \mathbb{C}$ such that $\alpha x_1 + \beta x_2 \in I$.
  (b) Show that $x_1 + x_3 \notin I$.
  (c) In fact, show there is a unique pair $\gamma, \delta \in \mathbb{C}$ such that $\gamma x_1 + \delta x_3 \in I$.