Exercise Sheet

for the **23/06/2023**

Exercise 1 Apply the Zero-Testing algorithm to $u(x) = \sqrt{1+x}$ and q for checking whether $q(u(x)) = 0$.

a) $q = 2uu' - 1$;
b) $q = 2uu' - 1 + x^{10}/10^{10}$;
c) $q = (2uu' - 1)u''^2 - xu + 1$.

Exercise 2 Try to find a root-separation bound for $p = xu' - cu$ where $c \in \mathbb{C}$ is a parameter.

Exercise 3 Compute the differential dimension polynomial of the differential ideals given by

a) $p = u'^2 - 1$;
b) $p = u^2u' - u$;
c) $p = u'^2 - u$.

Can we find the formal power series solutions, i.e., the differential counting polynomial?

Exercise 4 Compute the differential counting polynomial of the incompressible Navier-Stokes equations given by

$$\begin{aligned} p_1 &= u_t + uu_{x_1} + vu_{x_2} + wu_{x_3} + p_{x_1} - u_{x_1,x_1} - u_{x_2,x_2} - u_{x_3,x_3} = 0, \\ p_2 &= v_t + uv_{x_1} + vv_{x_2} + wv_{x_3} + p_{x_2} - v_{x_1,x_1} - v_{x_2,x_2} - v_{x_3,x_3} = 0, \\ p_3 &= w_t + uw_{x_1} + vw_{x_2} + ww_{x_3} + p_{x_3} - w_{x_1,x_1} - w_{x_2,x_2} - w_{x_3,x_3} = 0, \\ p_4 &= u_{x_1} + v_{x_2} + w_{x_3} = 0. \end{aligned}$$

where the Poisson pressure equation

$$p_5 = 2u_{x_2}v_{x_1} + 2u_{x_3}w_{x_1} + 2v_{x_3}w_{x_2} + u_{x_1}^2 + v_{x_2}^2 + w_{x_3}^2 + p_{x_1,x_1} + p_{x_2,x_2} + p_{x_3,x_3} = 0$$

is added to the system in order to obtain a regular differential chain (w.r.t. an orderly ranking where $u > v > w > p$ and $u_{x_1} > u_{x_2} > u_{x_3} > u_t$).

Exercise 5 Compute the input-output equation of the realization

$$\begin{cases} t_1' = t_1/t_2 \\ t_2' = t_1 \\ u = 2t_2 \end{cases} .$$

Exercise 6 Check whether the following input-output equations are realizable.

a) $p = u^2 + u'^2 - 1$;
b) $p = u' + yu^2 - 2y^3u - 2yy' + y^5$;
c) $p = (u' - yu)^3 + yu^2$;
d) $p = 2u^2u'^2 + u^2 + 2u'^2$.

For finding rational parametrizations of $\mathbb{V}(p)$ you might use the `Maple` command `algcurves:-parametrization`.

Exercise 7 Find the dynamical systems corresponding to the rational parametrization $\mathcal{P}$ of $\mathbb{V}(p)$ and decide with this whether $p = 0$ has a rational generic solution.

a) $p = u^2u' + u^2 + 2u'^2 + 8u + 8, \mathcal{P} = (\frac{t^2-8}{2t}, -\frac{t^2+8}{t^2})$;

b) $p = 27u^4 + u'^3, \mathcal{P} = (-t^3/27, -t^4/27)$;

c) $p = u^2u' + u^2 - u'^2 + 2u' - 1, \mathcal{P} = (\frac{-t^2+2}{t}, t^2 - 1)$;

In the negative case, can we find a local solution?

Exercise 8 Compute a local solution of $p = u^3u' - u'^3 + 2u$ with the initial value $u(0) = 0$. A local parametrization is given by $\mathcal{P} = (t^3/2, t + t^8/24 + t^{22}/41472 + \mathcal{O}(t^{29}))$ and can be found by the `Maple` command `algcurves:-puiseux`.

Exercise 9 Consider the differential equation $p = -8uu'^3 + 27u^2 + 4u'^2 - 4 = 0$. Compute a local solution with $u(0) = 0$ and check whether it is algebraic.

Exercise 10 Consider the differential equation $p = xu' - u/2 - x = 0$. Compute a local solution with $u(0) = 0$ and check whether it is algebraic.