TROPICAL DIFFERENTIAL ALGEBRAIC GEOMETRY EXERCISES SHEET.

CRISTHIAN E. GARAY-LÓPEZ

Conventions. Every algebraic structure will be commutative and associative. K will denote a field. Also $0 \in \mathbb{N}$.

If $f \in R[x_1, \ldots, x_n]$, say $f = \sum_I a_I x^I$, we will denote by ev_f the (evaluation) polynomial function $R^n \to R$ sending $p = (p_1, \ldots, p_n)$ to $\sum_I a_I p^I$.

1. Algebra

Exercise 1.1. *Let $\mathcal{P}(X)$ be the power set of a non-empty set. Show that the only homomorphism of commutative monoids $(\mathcal{P}(X), \cup, \emptyset) \longrightarrow (G, +, 0)$ with G a group, is the zero homomorphism.*

Exercise 1.2. *Let S be an idempotent semiring. Show that if $\sum_{i=1}^n a_i = 0$ holds in S, then $a_i = 0$ for all i.*

Exercise 1.3. *Let $\mathbb{B} = \{0 < 1\}$ be the Boolean semiring.*

(1) Show that $(\mathbb{B}[\![t, u]\!], +, \times, 0, 1)$ endowed with the usual operations of product and sum of formal power series is an idempotent semiring.

(2) Show that $\frac{\partial}{\partial t}, \frac{\partial}{\partial u} : \mathbb{B}[\![t, u]\!] \longrightarrow \mathbb{B}[\![t, u]\!]$ are semiring derivations, this is, they are linear operators that satisfy the product rule. Show that they commute under composition.

2. Tropical differential algebra

Let S be either a field of characteristic zero, or the Boolean semiring. Let $\mathbb{N}^m$ be the free monoid generated by the set of pairwise-commuting derivations $\{\frac{\partial}{\partial t_i} \,:\, i = 1, \ldots, n\}$ under composition of functions.

Recall that $S_{m,n} = S[\![t_1, \ldots, t_m]\!]\{x_1, \ldots, x_n\}$ and $\Theta : \mathbb{N}^m \times S_{m,n} \to S_{m,n}$ is the action that allows us to diferentiate differential polynomials and to define the twisted evaluation map $ev_{P,\Theta} : S[\![t_1, \ldots, t_m]\!]^n \to S[\![t_1, \ldots, t_m]\!]$ for $P \in S_{m,n}$.

Exercise 2.1 **(*).** *Let $P \in \mathbb{B}_{2,1} = \mathbb{B}[\![t, u]\!]\{x\}$. Find the conditions upon which $\varphi \in \mathbb{B}[\![t, u]\!]$ satisfies $ev_{P,\Theta}(\varphi) = P(\varphi) = 0$.*

Exercise 2.2 **(*).** *Let $Supp : K_{2,1} \to \mathbb{B}_{2,1}$ be the tropicalization map $\sum_i a_i M_i \mapsto \sum_i Supp(a_i) M_i$. Show that this is a non-Archimedean seminorm.*

3. Computations

Exercise 3.1. *Let $P \in \mathbb{B}[\![t]\!]\{x\}$ be the differential polynomial $P = x + x_1$. Show that*

$$Sol(P) = \{0\} \cup \{1 + t + \xi \,:\, \xi \in \mathbb{B}[\![t]\!]\}.$$

Exercise 3.2. *Let $P \in \mathbb{C}_{2,1} = \mathbb{C}[\![t, u]\!]\{x\}$ be the differential polynomial $P = tx_{(1,0)} + ux_{(0,1)} + (t^2 + u^3)$. Compute $Sol(P) \subset \mathbb{C}[\![t, u]\!]$.*

Hint. Suppose that $\varphi = \sum_{i,j} a_{i,j} t^i u^j \in Sol(P)$.

Exercise 3.3. *Let $P \in \mathbb{B}_{2,1} = \mathbb{B}[\![t, u]\!]\{x\}$ be the differential polynomial $P = tx_{(1,0)} + ux_{(0,1)} + (t^2 + u^3)$.*

(1) Show that $\varphi = t^2 + u^3 + tu \in \mathbb{B}[\![t, u]\!]$ belongs to $Sol(P)$.

(2) Moreover, show that

$$Sol(P) = \{\varphi = t^2 + u^3 + \xi \,:\, \xi \in \mathbb{B}[\![t, u]\!], \quad (1,0), (0,1), (0,2) \notin Supp(\xi)\}.$$

Exercise 3.4. *Consider the complex differential polynomial* $P = tx_{(1,0)} + ux_{(0,1)} + (t^2 + u^3)$.

1. *Show that if* $\varphi \in Sol(P) \subset \mathbb{C}[\![t,u]\!]$*, then* $Supp(\varphi) \in Sol(Supp(P))$*. See Exercise 3.2,*
2. *Consider the tropical differential polynomial* $Q = Supp(P)$ *(see Exercise 2.2). We know (Ex. 3.3 that* $\psi = t^2 + tu + u^3$ *is in* $Sol(Q)$*. Show that* ψ *can not be lifted to a complex solution.*

Exercise 3.5. *Let* $P \in \mathbb{C}_{1,1}$ *be the polynomial* $P = x - x_1$*, and let* $[P]$ *be its differential ideal.*

1. *Use the Fundamental Theorem to compute* $Sol(Supp([P]))$
2. *Compute the fiber* $Supp^{-1}(\varphi)$ *for every* $\varphi \in Sol(sp([P]))$*.*

Exercise 3.6. *Can you compute* $Sol(Supp([P]))$ *for* $x + x_1 = P \in \mathbb{C}_{1,1}$*?*

4. Vertex polynomials

For $m \geq 1$, we denote by $\mathbb{VB}[t_1, \dots, t_m]$ the set of vertex polynomials. Given $A, B \in \mathbb{VB}[t_1, \dots, t_m]$, $m \geq 1$ we define:

$$A \oplus B := V(A + B), \quad A \odot B := V(AB). \tag{1}$$

where $A + B$ and AB are computed in $\mathbb{B}[\![t_1, \dots, t_m]\!]$. See Exercise 1.3.

Exercise 4.1 (Case $m = 1$)**.** *Show that* $(\mathbb{VB}[t], \oplus, \odot) \cong (\mathbb{N} \cup \{\infty\}, min, +)$*, where min is considered with respect to the usual order in* $\mathbb{N} \cup \{\infty\}$*.*
Hint: Show $(\mathbb{VB}[t], \oplus, \odot) = (\mathbb{Z}_{\leq 0} \cup \{-\infty\}, max, +)$

In general we have

$$A \oplus B \subset V(A) \cup V(B), \quad A \odot B \subset V(A) + V(B), \tag{2}$$

but the contention may be proper.

Exercise 4.2. *Consider* $a = t^2u^3 + t^3u + t^5 = \{(2,3), (3,1), (5,0)\}$ *and* $b = u^4 + tu^3 + t^4u^2 = \{(0,4), (1,3), (4,2)\}$ *in* $a, b \in \mathbb{B}[\![t,u]\!]$*.*

1. *Show that* a *and* b *are vertex polynomials,*
2. *compute* $a \oplus b$ *and* $a \odot b$*, and show that the inclusion* (2) *is proper.*

5. Open problems for vertex polynomials (*)

The concept of vertex polynomials for $m > 1$ is much more complicated. We will now propose some open problems related to this concept.

Exercise 5.1 (Computation of vertex polynomials)**.** *Explicit computation / characterization of vertex polynomials of Newton polyhedra of clouds of points. Or computation of vertex sets of integral polyhedra of blocking type.*

Exercise 5.2 (Certificate for vertex polynomials)**.** *Is there a way to certify if a polynomial* $a \in \mathbb{B}[\![t_1, \dots, t_m]\!]$ *is a vertex polynomial?*

Exercise 5.3 (Effective computation of algebraic operations on vertex polynomials)**.** *Is there a way of computing the operations* (1)*?*

Exercise 5.4 (Certificate of tropical vanishing condition)**.** *Is there a way to certify whether or not a sum* $s = a_1 \oplus \cdots \oplus a_m$ *in* $\mathbb{VB}[t_1, \dots, t_m]$*vanishes tropically?*

References

[1] Ethan Cotterill, Cristhian Garay-López, Johana Luviano, *Exploring tropical differential equations.* To appear in Advances in Geometry. `https://arxiv.org/abs/2012.14067`
[2] Cristhian Garay-López, *Introducción a las ecuaciones diferenciales tropicales*. Mixba'al, Revista Metropolitana de Matemáticas. Vol.13, No.1, pp. 11-28. `www.doi.org/10.24275/uami/dcbi/mix/v13n1/cgaray`