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Applications of algebraic geometry in the field of Partial Differential Equations

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Conserved quantities are crucial in PDEs, defining the system behavior.

Conservation and energy estimats

Example

Consider the one-dimensional wave equation with periodic boundary conditions:

$$
\begin{aligned}
u_{tt} &= c^2 u_{xx}, \quad 0 \leq x \leq L, t > 0 \\
u(0,t) = u(L,t), &\quad u_x(0,t) = u_x(L,t), \quad t > 0.
\end{aligned}
\tag{1}
$$

Energy in the system is given by:

$$E(t) = \frac{1}{2}\int_0^L (u_t(x,t)^2 + c^2 u_x(x,t)^2)dx.$$

Conservation and energy estimats

$$\frac{d}{dt}E(t) = \int_0^L u_t(x,t)u_{tt}(x,t) + c^2 u_x(x,t)u_{xt}(x,t)dx.$$

Rewriting the first term using the PDE and integrating it by parts, we get:

$$\begin{aligned}
\frac{d}{dt}E(t) &= c^2[u_x(L,t)u_t(L,t) - u_x(0,t)u_t(0,t)] \\
&\quad - \int_0^L c^2 u_x(x,t)u_{tx}(x,t)dx + \int_0^L c^2 u_x(x,t)u_{xt}(x,t)dx \\
&= \int_0^L c^2 u_x(x,t)u_{xt}(x,t) - c^2 u_x(x,t)u_{xt}(x,t)dx \\
&= 0.
\end{aligned}$$

One diminssional heat equation

Consider the heat equation in one-dimension on some periodic spatial domain T^d.

$$u_t = \alpha u_{xx},$$

where $u(x,t) \in C^\infty$ represents the temperature and α is the thermal diffusivity of the material being modeled.
The integral quantity $\int_{T^d} u^2(x,t)dx$ represents the total energy of the system.

This quantity is dissipative in time.

Preserving them is vital for reliable numerical approximations.

Neglecting conservation leads to unreliable, meaningless solutions.

Building conservative schemes are curtal.

Testing for conservation

Consider

- $\mathcal{J}[u] = \sum_{n \in T^d} F(u((n,t)+e_1), \ldots, u((n,t)+e_k))$.
- The scheme $S = \{E_1[u] = 0, E_2[u] = 0, \ldots\}$

Testing if $\mathcal{J}$ is conserved for S:

Does the discrete time derivative of $\mathcal{J}$ given by

$$\mathcal{D} = \mathcal{J}[u](t+1) - \mathcal{J}[u](t),$$

remain constant over time?

The deduction of conservation as an ideal membership problem

Does $\mathcal{D}$ belong to the ideal generated by S?

We first shift from difference to polynomial settings where the ideal generated by the scheme $\mathcal{I} = \langle S \rangle$ is contained in the ring:

$$\mathcal{R} = \mathcal{R}[u((n,t)+e_1), u((n,t)+e_2), \ldots].$$

Step 1: We set up the discrete time derivative.

Step 2: We shift from difference to polynomial settings by translating the scheme such that the variables in the translated scheme $\mathcal{I}_0$ cover the variables of the discrete time derivative.

Step 3: We generate a Gröbner bases $\mathcal{G}$ of the ideal $\mathcal{I}_0$

Step 4: Divide $\mathcal{D}$ by $\mathcal{G}$, return the remainder.

Step 5: Compute the discrete partial variational derivative of the remainder.

Step 6: We repeat steps 2 to 4 on the result of the previous step.

Step 7: If the result is zero, then $\mathcal{J}$ is conserved on the solutions of the scheme S.

Previous approaches lack a systematic methodology for effectively handling parameters in PDEs.

This is a critical issue as parameters significantly impact the conservation of quantities.

Example

The energy $\int u^2$ in heat equation $u_t = au_{xx}$, is not conserved unless $a = 0$

Setting: Parametric Polynomial Rings

Definition

$\mathcal{K}[a_1, .., a_s][x_1, ..., x_n]$ is a parametric ring.

$x_1, ..., x_n$ are the independent variables.

Coefficients are polynomials in the ring $\mathcal{K}[a_1, .., a_s]$.

For simplicity, we denote this ring by $\mathcal{K}[A][X]$.

Specializations (aka, choice of parameters)

Definition

A specialization is a ring homomorphism.

$\sigma : \mathcal{K}[A][X] \to K[X]$,

given by a polynomial evaluation on the parameters and an identity on the variables.

Comprehensive Gröbner System (CGS)

Definition

$\mathcal{I}$ is a finitely generated parametric ideal.

The family $\{A_1, \ldots, A_k\}$ is a partition of K^m.

$\{G_1, \ldots, G_k\}$ is a family of subsets of $\mathcal{K}[A][X]$.
The system,

$$\begin{cases} G_1 & A_1 \\ \vdots & \vdots \\ G_k & A_k \end{cases}$$

is a CGS of I, if for all specializations
$\sigma_i \in A_i, \sigma_i(G_i)$ is a Gröbner basis of the specialized ideal $\sigma_i(I)$.

Example

Consider $\mathcal{I} \subset \mathbb{R}[a, b][x, y]$ generated by

$$\mathcal{F} = \left\{ax^2 + \left(a^2 - 2ab\right) xy, b^2y^2 - x\right\}.$$

The CGS of $\mathcal{I}$ with a Lex order is:

$$\begin{cases} \left\{a^2b^2y^3 + ab^4y^4 - 2ab^3y^3, x - b^2y^2\right\} & ab \neq 0 \\ \{x\} & b = 0 \\ \left\{b^2y^2 - x\right\} & a = 0 \wedge b \neq 0. \end{cases}$$

Existence of CGSs

Theorem
Let $\mathcal{I} \subseteq \mathcal{K}[A][X]$. The corresponding comprehensive Gröbner System exists.

Issues of Neglecting Parameters

- In general, $\sigma(\mathcal{G})$ is not necessarily a Gröbner basis.

Example

Let $\mathcal{I} = \langle ax^2 - y, -2 + bxy \rangle$. Consider the specialization $\sigma \in \mathbb{R}^2$ given by $P \mapsto P(0,1)$.

The set $\sigma(\mathcal{G}) = \{y^3, -y^2, xy - 2\}$

is not a Gröbner basis.

- The algorithm to resolve the ideal membership problem using Gröbner basis can not be used for all specializations.

The limitations disscused above highlight why we cannot rely solely on the use of Gröbner basis.

Ideal Membership Problem for Parametric Ideals

Definition

$f \in \mathcal{I}$, iff there exists a specialization $\sigma \in K^m$ such that $\sigma(f) \in \sigma(\mathcal{I})$.

Solving Parametric Ideal Membership problem using CGSs

Now, to determine if $f \in \mathcal{I}$, we consider the CGS of $\mathcal{I}$

$$\begin{cases} G_1 & A_1 \\ \vdots & \vdots \\ G_k & A_k. \end{cases}$$

We check if there exists $\sigma_i \in A_i$ such that $f \in \sigma_i(\mathcal{I})$ for each $i \in \{1, .., k\}$.

Example

Let $\mathcal{F} = \{a^2x - y, -b^2xy^2 - x\}$ and let $\mathcal{I} \subset \mathbb{R}[a, b][x, y]$ be the ideal generated by $\mathcal{F}$. We will determine if $f(x, y) = -x + y^2 \in \mathcal{I}$.

Discrete Parametric Rings

We consider:

- $\mathcal{I}$ generated by the parametric polynomial finite difference scheme $H = \{E_1[u] = 0, \ldots, E_s[u] = 0\}$.
- $\mathcal{K}[A][\tilde{X}]$ where $\tilde{X}$ is the set of the independent variables $u((n,t) + e_i)$

$$\mathcal{K}[A][u((n,t)+e_1), u((n,t)+e_2), \ldots, u((n,t)+e_k)].$$

Setting:

Consider:

- $\mathcal{J}[u] = \sum_{n \in T^d} F(u((n,t)+e_1), \dots, u((n,t)+e_k))$.
- The parametric scheme $H = \{E_1[u] = 0, \dots, E_s[u] = 0\}$

Step 1: We set up the discrete time derivative :

$$\mathcal{D} = \mathcal{J}[u](t+1) - \mathcal{J}[u](t).$$

Step 2: We translate the scheme to cover all independent variables occurring in the discrete time derivative.

Step 3: We compute the corresponding CGS of the ideal $\mathcal{I}_0$ generated by the translated scheme

Step 4: We reduce $\mathcal{D}$ by the CGS of the translated scheme to obtain a piecewise remainder

$$\begin{cases} r_1 & A_1 \\ \vdots & \vdots \\ r_k & A_k. \end{cases}$$

Step 5: We compute discrete partial variational derivative of all non-zero remainders r_i.

Step 6: We repeat steps 2 to 4 for any non-zero discrete partial variational derivative from the previous step.

Step 7: We conclude conservation for the zero results of steps 4, 5, and 6.

Remarks

- We implement this algorithm in the function DiscreteConservedQOperator.
- This algorithm can identify conservation; however, it cannot provide definitive confirmation regarding the absence of conservation.

The One-dimensional Advection Equation

Consider,

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0, \quad 0 \leq x \leq L, t > 0$$

$$u(0,t) = u(L,t), \quad t > 0.$$

The total energy $\int_0^L u^2(x,t)dx$ is conserved under periodic boundary conditions.

Parametric Numerical Scheme

Consider the centered in space, forward in time finite difference scheme:

$$c(u(n+1,t)-u(n-1,t))+u(n,t+1)+\frac{1}{2}(-u(n-1,t)-u(n+1,t))=0.$$