2-decoupling-fractions

June 13, 2023

[1]:
```
from sympy import *
from DifferentialAlgebra import *
init_printing ()
```

From "Decoupling Multivariate Fractions", François Lemaire and Adrien Poteaux, CASC 2021

1 Decoupling example

Given a fraction F in Q(x,y), find (if they exist) c in Q, G in Q(x) and H in Q(y) such that F = c + G*H

[2]:
```
x, y = var ('x, y')
Fbar = 1 - 1/(1+x)*1/(1+y)
Fbar = simplify (Fbar)
Fbar
```

[2]:

$$\frac{(x+1)(y+1)-1}{(x+1)(y+1)}$$

[3]:
```
c, F, G, H = indexedbase('c, F, G, H')
```

If c exists, it can be computed from F by a formula which can be obtained by differential elimination

Ranking: (G,H) > c > F

[6]:
```
R = DifferentialRing (derivations = [x,y], blocks = [[G,H],c,F], notation =␣
↪'jet')
```

[7]:
```
syst = [ Eq (F, c + G*H), Eq(G[y],0), Eq(H[x],0), Eq(c[x],0), Eq(c[y],0), Ne␣
↪(G[x],0), Ne (H[y],0) ]
syst
```

[7]:

$$[F = GH + c,\ G_y = 0,\ H_x = 0,\ c_x = 0,\ c_y = 0,\ G_x \neq 0,\ H_y \neq 0]$$

[8]:
```
ideal = R.RosenfeldGroebner (syst)
ideal
```

[8]:
```
[regular_differential_chain]
```

[9]:
```
A = ideal[0]
A.equations(solved=True)
```

[9]:

$$\left[F_{x,y,y} = \frac{F_{x,y}F_{y,y}}{F_y},\ F_{x,x,y} = \frac{F_{x,x}F_{x,y}}{F_x},\ c = \frac{F_{x,y}F - F_xF_y}{F_{x,y}},\ G = \frac{F_xF_y}{F_{x,y}H},\ H_y = \frac{F_{x,y}H}{F_x},\ H_x = 0\right]$$

[10]:
```
cbar = Fbar - R.differentiate(Fbar,x)*R.differentiate(Fbar,y)/R.
 ↪differentiate(Fbar,x*y)
cbar = simplify (cbar)
cbar
```

[10]:

$$1$$

[11]:
```
Fhat = Fbar - cbar
Gbar = simplify (Fhat.subs ({y:0}))
Gbar
```

[11]:

$$-\frac{1}{x+1}$$

[12]:
```
Hbar = simplify (Fhat / Gbar)
Hbar
```

[12]:

$$\frac{1}{y+1}$$

[13]:
```
simplify (Fbar - (cbar + Gbar*Hbar))
```

[13]:

$$0$$

2 Proof that the four cases are exclusive

The four cases are: 1. F = G + H 2. F = c + G*H 3. F = c + 1/(G+H) 4. F = c + d/(1 + G*H)

The proof consists in writing systems requiring two different decompositions for the same F and prove inconsistency in each case

[14]:
```
x, y = var ('x, y')
F = indexedbase ('F')
c, G, H = indexedbase('c, G, H')
d, e, P, Q = indexedbase('d, e, P, Q')
```

[15]:
```
R = DifferentialRing (derivations = [x,y],
                      blocks = [[F, c, G, H, d, e, P, Q]],
                      notation = 'jet')
```

[16]:
```
syst = [ Eq (F, G + H), Eq(G[y],0), Eq(H[x],0), Ne (G[x],0), Ne (H[y],0),
         Eq (F, d + P*Q), Eq(P[y],0), Eq(Q[x],0), Eq(d[x],0), Eq(d[y],0),
            Ne (P[x],0), Ne (Q[y],0)]
```

```
syst
```

[16]: $\left[F = G + H,\ G_y = 0,\ H_x = 0,\ G_x \neq 0,\ H_y \neq 0,\ F = PQ + d,\ P_y = 0,\ Q_x = 0,\ d_x = 0,\ d_y = 0,\ P_x \neq 0,\ Q_y \neq 0\right.$

[17]:
```
R.RosenfeldGroebner(syst)
```

[17]: []

[16]:
```
syst = [ Eq (F, G + H), Eq(G[y],0), Eq(H[x],0), Ne (G[x],0), Ne (H[y],0),
         Eq (F, d + 1/(P + Q)), Eq(P[y],0), Eq(Q[x],0), Eq(d[x], 0), Eq(d[y],
 ↪0),
            Ne (P[x],0), Ne (Q[y],0) ]
syst
```

[16]: $\left[F = G + H,\ G_y = 0,\ H_x = 0,\ G_x \neq 0,\ H_y \neq 0,\ F = d + \frac{1}{P+Q},\ P_y = 0,\ Q_x = 0,\ d_x = 0,\ d_y = 0,\ P_x \neq 0,\ Q_y\right.$

[17]:
```
R.RosenfeldGroebner(syst)
```

[17]: []

[18]:
```
syst = [ Eq (F, G + H), Eq(G[y],0), Eq(H[x],0), Ne (G[x],0), Ne (H[y],0),
         Eq (F, d + e/(1 + P*Q)), Eq(P[y],0), Eq(Q[x],0), Eq(d[x], 0),
            Eq(d[y], 0), Eq(e[x], 0), Eq(e[y], 0), Ne (P[x],0), Ne (Q[y],0) ]
syst
```

[18]: $\left[F = G + H,\ G_y = 0,\ H_x = 0,\ G_x \neq 0,\ H_y \neq 0,\ F = d + \frac{e}{PQ+1},\ P_y = 0,\ Q_x = 0,\ d_x = 0,\ d_y = 0,\ e_x = 0,\ e_y\right.$

[19]:
```
R.RosenfeldGroebner(syst)
```

[19]: []

[20]:
```
syst = [ Eq (F, c + G*H), Eq(G[y],0), Eq(H[x],0), Eq(c[x],0), Eq(c[y],0), Ne
 ↪(G[x],0), Ne (H[y],0),
         Eq (F, d + 1/(P + Q)), Eq(P[y],0), Eq(Q[x],0), Eq(d[x], 0), Eq(d[y],
 ↪0),
            Ne (P[x],0), Ne (Q[y],0) ]
syst
```

[20]: $\left[F = GH + c,\ G_y = 0,\ H_x = 0,\ c_x = 0,\ c_y = 0,\ G_x \neq 0,\ H_y \neq 0,\ F = d + \frac{1}{P+Q},\ P_y = 0,\ Q_x = 0,\ d_x = 0,\ d_y\right.$

[21]:
```
R.RosenfeldGroebner(syst)
```

[21]: []

[22]:
```
syst = [ Eq (F, c + G*H), Eq(G[y],0), Eq(H[x],0), Eq(c[x],0), Eq(c[y],0), Ne
 ↪(G[x],0), Ne (H[y],0),
         Eq (F, d + e/(1 + P*Q)), Eq(P[y],0), Eq(Q[x],0), Eq(d[x], 0),
```

```
            Eq(d[y], 0), Eq(e[x], 0), Eq(e[y], 0), Ne (P[x],0), Ne (Q[y],0) ]
syst
```

[22]:

$$\left[F = GH + c,\ G_y = 0,\ H_x = 0,\ c_x = 0,\ c_y = 0,\ G_x \neq 0,\ H_y \neq 0,\ F = d + \frac{e}{PQ+1},\ P_y = 0,\ Q_x = 0,\ d_x = 0,\ d_y\right.$$

[23]:

```
R.RosenfeldGroebner(syst)
```

[23]:

[]

[24]:

```
syst = [Eq (F, c + 1/(G + H)), Eq(G[y],0), Eq(H[x],0), Eq(c[x], 0), Eq(c[y],
 ↪0), Ne (G[x],0), Ne (H[y],0),
        Eq (F, d + e/(1 + P*Q)), Eq(P[y],0), Eq(Q[x],0), Eq(d[x], 0),
            Eq(d[y], 0), Eq(e[x], 0), Eq(e[y], 0), Ne (P[x],0), Ne (Q[y],0) ]
syst
```

[24]:

$$\left[F = c + \frac{1}{G+H},\ G_y = 0,\ H_x = 0,\ c_x = 0,\ c_y = 0,\ G_x \neq 0,\ H_y \neq 0,\ F = d + \frac{e}{PQ+1},\ P_y = 0,\ Q_x = 0,\ d_x = 0\right.$$

[25]:

```
R.RosenfeldGroebner(syst)
```

[25]:

[]

[ ]: