Differential Algebra — Exercises

François Boulier
University of Lille, CRIStAL, France

CIMPA School: *Algebraic and Tropical Methods for Solving Differential Equations*

June 14, 2023

Comment

The exercises are not supposed to be difficult

They are mostly designed to incite you to seek the relevant information in the slides

Lecture 1

Derivative of a fraction

Using axioms of derivations, prove that

$$\left(\frac{a}{b}\right)' = \frac{\dot{a}\, b - a\, \dot{b}}{b^2}$$

Derivative of a fraction

Using axioms of derivations, prove that

$$\left(\frac{a}{b}\right)' \quad = \quad \frac{\dot{a}\, b - a\, \dot{b}}{b^2}$$

Hint:

$$\left(\frac{a}{b}\right) \times b \quad = \quad a$$

Rankings

A *ranking* is a total order on the infinite set ΘY of the derivatives which satisfies the two following axioms, for all derivatives $v, w \in \Theta Y$ and every derivation δ:

1. $v \leq \theta v$ and
2. $v < w \Rightarrow \theta v < \theta w$.

How many rankings exist in the case of a single differential indeterminate and a single derivation?

Rankings

A *ranking* is a total order on the infinite set ΘY of the derivatives which satisfies the two following axioms, for all derivatives $v, w \in \Theta Y$ and every derivation δ:

1. $v \leq \theta v$ and
2. $v < w \Rightarrow \theta v < \theta w$.

Does there exist a relationship between rankings and the admissible orderings of the Gröbner bases theory?

Differential Polynomials

$$p = y_1^2\,\dot{y}_1^3 + 6\dot{y}_1\,\ddot{y}_1^2$$

What is the leading derivative of p, its initial and its separant?

Differential Polynomials

$$p \quad = \quad y_1^2\,\dot{y}_1^3 + 6\dot{y}_1\,\ddot{y}_1^2$$

Differentiate p. What about the initial of $\dot{p}$?

Differential Ideals

Prove $\dot{y}^3 \in [y^2]$

Differential Ideals

What can be said about the perfect differential ideal $\{y^2\}$?

Differential Ideals

Let Σ be a set of differential polynomials.

True or false? $1 \in [\Sigma] \Leftrightarrow 1 \in \{\Sigma\}$

Differential Ideals

A differential ideal $\mathfrak{A}$ is defined as an ideal such that $p \in \mathfrak{A} \Rightarrow \dot{p} \in \mathfrak{A}$

Let Σ be a set of differential polynomials.

Is it obvious that $[\Sigma]$ is the ideal (in the non differential sense) generated by $\Theta\Sigma$ which contains the elements of Σ and all their dervatives up to any order?

Differential Ideals

It can be proved that

$$\{\dot{y}^2 - 4\,y\} \quad = \quad \{\dot{y}^2 - 4\,y,\ \ddot{y} - 2\} \cap \{y\}$$

where the right hand side differential ideals are prime.
Prove that none of the right hand side differential ideals is included in the other one
Can we deduce that the left hand side differential ideal is not prime?

Annihilating ODE

We are looking for a differential polynomial p which is annihilated by the following expression

$$\text{expr} \quad = \quad x^3 + x^{\frac{4}{3}}$$

Set up a differential elimination problem which permits to compute p. Which system? Which ranking?

Input-Output Equation

Lotka-Volterra prey-predator model is

$$\begin{aligned} \dot{y}_1 &= \alpha\, y_1 - \beta\, y_1\, y_2 \\ \dot{y}_2 &= \mu\, y_1\, y_2 - \gamma\, y_2 \end{aligned}$$

The differential indeterminates y_1 and y_2 represent the densities of populations of two animal species. Greek letters are parameters.

A differential polynomial has been obtained by differential elimination over the model equations. It has been divided by its initial. The result is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma\, y_1 - \frac{\mu\, y_1^2}{2}\right) + \alpha\,\mu\, y_1^2 - \alpha\,\gamma\, y_1 \quad = \quad \frac{\dot{y}_1^2}{y_1} - \ddot{y}_1$$

Which ranking should be provided to the differential elimination software to obtain such an equation?

Input-Output Equation

Lotka-Volterra prey-predator model is

$$\begin{aligned}\dot{y}_1 &= \alpha\, y_1 - \beta\, y_1\, y_2\\ \dot{y}_2 &= \mu\, y_1\, y_2 - \gamma\, y_2\end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\gamma\, y_1 - \frac{\mu\, y_1^2}{2}\right) + \alpha\,\mu\, y_1^2 - \alpha\,\gamma\, y_1 \;=\; \frac{\dot{y}_1^2}{y_1} - \ddot{y}_1$$

Assume the knowledge of a function $y_1(t)$ satisfying the equation, so that any differential fraction in $\mathscr{F}<y>$ can be evaluated to numbers, at $t = t_1, \ldots, t_9$

Write an overdetermined linear system $A\,x = b$, where A is matrix and b a vector of numbers, stating the equation is satisfied at $t_1, \ldots, t_m$

What are the dimension of A? the parameter blocks in x?

Assume A has full rank so that $A\,x = b$ can be solved by linear least squares. Is it possible to recover the values of the model parameters?

The differential $\mathscr{F}$-algebra of differential power series in one differential indeterminate y is denoted

$$\mathscr{F}\{\{y\}\}$$

An element of this algebra is a possibly infinite sum of terms of the form

$$c_i\, t_i$$

where the $c_i \in \mathscr{F}$ and the t_i are power products of y and its derivatives up to any order

Draw a picture of Differential Power Serious

Partial Remainders

Compute the partial remainder of $\ddot{y}$ with respect to

$$p \quad = \quad \dot{y}^2 - 4\,y$$

How to relate this computation to the computation of formal power series solutiion of p?

Consider the differential polynomial

$$p \quad = \quad y\,\dot{y}^2 + y - 1$$

Compute p_0, p_1, p_2 defined by $p_i = p^{(i)}(y_0, y_1, y_2, y_3)$

Check that, for $y_0, y_1, y_2, y_3 = 1, 2, -2, -1$ we have $p_0, p_1, p_2 = 4, 2, -19$

Use the information to compute $p(\bar{\bar{y}}) \mod x^4$ where

$$\bar{\bar{y}} \quad = \quad y_0 + y_1\,x + y_2\,\frac{x^2}{2} + \cdots$$

Check

Linear ODE with Constant Coefficients

Assume p is a linear differential equation in one differential indeterminate and constant coefficients

Does the general method provided during the lecture simplify?

Partially Autoreduced and Coherent

Provide an example of a triangular system of ordinary differential polynomials the elements of which are not pairwise partially reduced

Provide an example of a triangular system of partial differential polynomials which is not coherent

In each case, provide a ranking

Inconsistent Systems

Provide an example of a triangular system A of differential polynomials, the elements of which are pairwise partially reduced, which is coherent but such that the following system has no solution

$$A = 0, \qquad H_A \neq 0.$$

Provide an example of a triangular system A of differential polynomials, the elements of which are pairwise partially reduced, which is coherent but such that the following system has no solution

$$A = 0, \qquad H_A \neq 0.$$

Hint: what if some element of A has multiple factors?

Existence Problem of Formal Power Series Solutions

Let Σ be a differential polynomial system

Among the two following problems, which one is equivalent to the problem of deciding if $1 \in \{\Sigma\}$?

1. The existence problem of formal power series solutions for prescribed initial values

2. The existence problem of formal power series solutions for some, unspecified, initial values

Is the problem of deciding if $1 \in \{\Sigma\}$ algorithmic?

Singer's Example

In which cases the following system has a nonzero formal power series solution?

$$\begin{aligned} x\,\dot{y} &= \alpha\, y \\ \dot{\alpha} &= 0 \end{aligned}$$

Hilbert's tenth problem: provide a general algorithm which, for any given Diophantine equation (a polynomial equation with integer coefficients and a finite number of unknowns), can decide whether the equation has a solution with all unknowns taking integer values

In 1970, Yuri Matiyasevich proved that no such algorithm exists whenever the number of unknowns is greater than or equal to 9

Using these informations, prove that there does not exist any algorithm which decides if a given arbitrary system of ordinary differential polynomials has a nonzero formal power series solution

Lecture 3

The Ideal $(A) : I_A^\infty$

Prove that $x_2 \in (A) : I_A^\infty$ where $A = x_1 \, x_2$

The Ideal $(A) : I_A^\infty$

True of false? A triangular set A is a basis of the ideal $(A) : I_A^\infty$

If true, why; if false, provide a counter-example

The Ideal $(A) : I_A^\infty$

Give a technique which permits to compute a basis of $(A) : I_A^\infty$

The Ideal $(A) : I_A^\infty$

Recall the definition of a zerodivisor modulo an ideal $\mathfrak{A}$

Prove that the initials of A are regular ($=$ non zerodivisors) modulo $(A) : I_A^\infty$

Regular Chains

Is the following system a regular chain?

$$A \left\{ \begin{array}{lcl} (x_1^2 - t_1^2)\, x_2 - x_1 + t_2 & = & 0 \\ x_1\,(x_1 - t_1) & = & 0 \end{array} \right.$$

Is the following system a regular chain?

$$A \left\{ \begin{array}{lcl} (t_1^2 - t_1)\, x_2 - x_1 + t_2 & = & 0 \\ x_1\, (x_1 - t_1) & = & 0 \end{array} \right.$$

Is the following system a regular chain?

$$A \begin{cases} (t_1^2 - t_1)\, x_2 - x_1 + t_2 & = & 0 \\ x_1\,(x_1 - t_1) & = & 0 \end{cases}$$

Hint: use the equivalence theorem for regular chains

Is the following system a squarefree regular chain?

$$A \left\{ \begin{array}{lcl} {x_2}^2 + x_1^2 - t_1 & = & 0 \\ {x_1}^2 - t_1 & = & 0 \end{array} \right.$$

Regular Differential Chains

Is the following system a regular differential chain?

$$A \left\{ \begin{array}{lcl} \dot{y}_2^{\,2} - \ddot{y}_1 & = & 0 \\ \dot{y}_1^{\,3} - y_1 + 1 & = & 0 \end{array} \right.$$

Regular Differential Chains

The following system corresponds to Euler equations for an incompressible fluid in two dimensions. The differential indeterminates v^1 and v^2 are the two components of the speed of the fluid. The differential indeterminate p is the pression. The three derivation operators are $\partial/\partial x_1$, $\partial/\partial x_2$ (the two space variables) and $\partial/\partial t$. Derivatives are denoted as follows: v^2_1 stands for $\partial/\partial x_1\, v^2$ and v^1_t stands for $\partial/\partial t\, v^1$

$$A \left\{ \begin{array}{lcl} v^1_t + v^1\, v^1_1 + v^2\, v^1_2 + p_1 & = & 0 \\ v^2_t + v^1\, v^2_1 + v^2\, v^2_2 + p_2 & = & 0 \\ v^1_1 + v^2_2 & = & 0 \end{array} \right.$$

This system is not coherent because one Δ-polynomial is not reduced to zero by A. Which Δ-polynomial?

Is this system a regular differential chain anyway?

Regular Differential Chains

The following system is a regular differential chain The differential indeterminates are u and v. The two derivation operators are $\partial/\partial x$ and $\partial/\partial y$

$$A \begin{cases} v_{xx} - u_x & = & 0 \\ 4\,u\,v_y + u_x\,u_y - u_x\,u_y\,u & = & 0 \\ {u_x}^2 - 4\,u & = & 0 \\ {u_y}^2 - 2\,u & = & 0 \end{cases}$$

Prove that $v_{xx} - u \notin [A] : h_A^\infty$

Regular Differential Chains

The following system is a regular differential chain The differential indeterminates are u and v. The two derivation operators are $\partial/\partial x$ and $\partial/\partial y$

$$A \begin{cases} v_{xx} - u_x & = & 0 \\ 4\,u\,v_y + u_x\,u_y - u_x\,u_y\,u & = & 0 \\ {u_x}^2 - 4\,u & = & 0 \\ {u_y}^2 - 2\,u & = & 0 \end{cases}$$

Prove that $v_x - u \notin [A] : H_A^\infty$

Hint: use the equivalence theorem for regular differential chains