Differential Algebra — Exercises

The exercises are not supposed to be difficult. They are mostly designed to incite you to seek the relevant information in the slides.

1 Lecture 1

Question 1. Using[1] axioms of derivations, prove that

$$\left(\frac{a}{b}\right)' = \frac{\dot{a}\,b - a\,\dot{b}}{b^2}$$

A *ranking* is a total order on the infinite set ΘY of the derivatives which satisfies the two following axioms, for all derivatives $v, w \in \Theta Y$ and every derivation δ:

1. $v \leq \delta v$ and

2. $v < w \Rightarrow \delta v < \delta w$.

Question 2. How many rankings exist in the case of a single differential indeterminate and a single derivation?

Question 3. Does there exist a relationship between rankings and the admissible orderings of the Gröbner bases theory?

Question 4. What is the leading derivative of p, its initial and its separant?

$$p = y_1^2\,\dot{y}_1^3 + 6\dot{y}_1\,\ddot{y}_1^2$$

Question 5. Differentiate p. What about the initial of $\dot{p}$?

Question 6. Prove $\dot{y}^3 \in [y^2]$

Question 7. What can be said about the perfect differential ideal $\{y^2\}$?

Question 8. Let Σ be a set of differential polynomials. True or false? $1 \in [\Sigma] \Leftrightarrow 1 \in \{\Sigma\}$

[1] Hint:

$$\left(\frac{a}{b}\right) \times b = a$$

Question 9. A differential ideal $\mathfrak{A}$ is defined as an ideal such that $p \in \mathfrak{A} \Rightarrow \dot{p} \in \mathfrak{A}$. Is it obvious that $[\Sigma]$ is the ideal (in the non differential sense) generated by $\Theta\Sigma$ which contains the elements of Σ and all their derivatives up to any order?

Question 10. It can be proved that

$$\{\dot{y}^2 - 4\,y\} \quad = \quad \{\dot{y}^2 - 4\,y,\ \ddot{y} - 2\} \cap \{y\}$$

where the right hand side differential ideals are prime. Prove that none of the right hand side differential ideals is included in the other one. Can we deduce that the left hand side differential ideal is not prime?

Question 11. We are looking for a differential polynomial p which is annihilated by the following expression (computations at the end of the document):

$$\mathrm{expr} \quad = \quad x^3 + x^{\frac{4}{3}}$$

Set up a differential elimination problem which permits to compute p. Which system? Which ranking?

— ∘ ◯ ∘ —

Lotka-Volterra prey-predator model is

$$\begin{aligned} \dot{y}_1 &= \alpha\,y_1 - \beta\,y_1\,y_2 \\ \dot{y}_2 &= \mu\,y_1\,y_2 - \gamma\,y_2 \end{aligned}$$

The differential indeterminates y_1 and y_2 represent the densities of populations of two animal species. Greek letters are parameters.

A differential polynomial has been obtained by differential elimination over the model equations (see details at the end of this document). It has been divided by its initial. The result is

$$\frac{\mathsf{d}}{\mathsf{d}t}\left(\gamma\,y_1 - \frac{\mu\,y_1^2}{2}\right) + \alpha\,\mu\,y_1^2 - \alpha\,\gamma\,y_1 \quad = \quad \frac{\dot{y}_1^2}{y_1} - \ddot{y}_1$$

Question 12. Which ranking should be provided to the differential elimination software to obtain such an equation?

Assume the knowledge of a function $y_1(t)$ satisfying the equation, so that any differential fraction in $\mathscr{F}{<}y_1{>}$ can be evaluated to numbers, at (say) $t = t_1, \ldots, t_9$.

Question 13. Write an overdetermined linear system $A\,x = b$, where A is matrix and b a vector of numbers, stating the equation is satisfied at $t_1, \ldots, t_9$. What are the dimension of A? the parameter blocks in x?

Question 14. Assume A has full rank so that $A\,x = b$ can be solved by linear least squares. Is it possible to recover the values of the model parameters?

2 Lecture 2

The differential $\mathscr{F}$-algebra of differential power series in one differential indeterminate y is denoted

$$\mathscr{F}\{\{y\}\}$$

An element of this algebra is a possibly infinite sum of terms of the form

$$c_i\, t_i$$

where the $c_i \in \mathscr{F}$ and the t_i are power products of y and its derivatives up to any order

Question 15. Draw a picture of Differential Power Serious.

Question 16. Compute the partial remainder of $\ddot{y}$ with respect to

$$p \;=\; \dot{y}^2 - 4\,y$$

How to relate this computation to the computation of formal power series solution of p?

— ∘ ◯ ∘ —

Question 17. Compute[2] p_0, p_1, p_2 defined by $p_i = p^{(i)}(y_0, y_1, y_2, y_3)$

$$p \;=\; y\,\dot{y}^2 + y - 1$$

Question 18. Check that, for $y_0, y_1, y_2, y_3 = 1, 2, -2, -1$ we have $p_0, p_1, p_2 = 4, 2, -38$

Question 19. Use the information to compute $p(\bar{\bar{y}}) \mod x^3$ where

$$\bar{\bar{y}} \;=\; y_0 + y_1\,x + y_2\,\frac{x^2}{2} + \cdots$$

Check

Question 20. Assume p is a linear differential equation in one differential indeterminate and constant coefficients. Does the general method provided during the lecture simplify?

[2]Computation details at the end of this document.

Question 21. Provide an example of a triangular system of ordinary differential polynomials the elements of which are not pairwise partially reduced.

Provide an example of a triangular system of partial differential polynomials which is not coherent.

In each case, provide a ranking.

Question 22. Provide an example of a triangular system A of differential polynomials, the elements of which are pairwise partially reduced, which is coherent but such that the following system has no solution[3]

$$A = 0, \qquad H_A \neq 0.$$

— ∘ ◯ ∘ —

Let Σ be a differential polynomial system

Question 23. Among the two following problems, which one is equivalent to the problem of deciding if $1 \in \{\Sigma\}$?

1. The existence problem of formal power series solutions for prescribed initial values

2. The existence problem of formal power series solutions for some, unspecified, initial values

Question 24. Is the problem of deciding if $1 \in \{\Sigma\}$ algorithmic?

— ∘ ◯ ∘ —

Hilbert's tenth problem: provide a general algorithm which, for any given Diophantine equation (a polynomial equation with integer coefficients and a finite number of unknowns), can decide whether the equation has a solution with all unknowns taking integer values.

In 1970, Yuri Matiyasevich proved that no such algorithm exists whenever the number of unknowns is greater than or equal to 9.

Question 25. In which cases the following system has a nonzero formal power series solution?

$$\begin{aligned} x\,\dot{y} &= \alpha\, y \\ \dot{\alpha} &= 0 \end{aligned}$$

[3]Hint: what if some element of A has multiple factors?

Question 26. Prove that there does not exist any algorithm which decides if a given arbitrary system of ordinary differential polynomials has a nonzero formal power series solution.

3 Lecture 3

Question 27. Prove that $x_2 \in (A) : I_A^\infty$ where $A = x_1\, x_2$

Question 28. True of false? A triangular set A is a basis of the ideal $(A) : I_A^\infty$. If true, why; if false, provide a counter-example.

Question 29. Give a technique which permits to compute a basis of $(A) : I_A^\infty$.

Question 30. Recall the definition of a zerodivisor modulo an ideal $\mathfrak{A}$. Prove that the initials of A are regular (= non zerodivisors) modulo $(A) : I_A^\infty$.

Question 31. Is the following system a regular chain?

$$A \left\{ \begin{array}{l} (x_1^2 - t_1^2)\,\mathbf{x_2} - x_1 + t_2 \\ \mathbf{x_1}\,(\mathbf{x_1} - t_1) \end{array} \right.$$

Question 32. Is[4] the following system a regular chain?

$$A \left\{ \begin{array}{l} (t_1^2 - t_1)\,\mathbf{x_2} - x_1 + t_2 \\ \mathbf{x_1}\,(\mathbf{x_1} - t_1) \end{array} \right.$$

Question 33. Is the following system a squarefree regular chain?

$$A \left\{ \begin{array}{l} \mathbf{x_2}^2 + x_1^2 - t_1 \\ \mathbf{x_1}^2 - t_1 \end{array} \right.$$

Question 34. Consider the following system

$$A \left\{ \begin{array}{l} t_1\,\mathbf{x_3}^2 - t_1 - x_2^2 \\ \mathbf{x_2}^2 - t_1^2 \\ x_1 \end{array} \right.$$

Compute a decomposition of the radical of the ideal (A) as an intersection of ideals defined by regular chains by applying the following strategy (computations detailed at the end of this document):

1. By splitting cases decompose the solution set of $A = 0$ into systems of the form $A_i = 0,\ I_{A_i} \neq 0$ where the A_i are regular chains.

[4]Hint: use the equivalence theorem for regular chains

2. Using the Theorem of Zeros (Nullstellensatz), transform this decomposition as a representation of the radical of the ideal (A) as an intersection of the radicals of the ideals defined by the regular chains A_i.

3. Prove that the regular chains A_i are squarefree. How does this information simplify the above formula?

Question 35. Is the following system a regular differential chain?

$$A \left\{ \begin{array}{l} \dot{\mathbf{y}}_{\mathbf{2}}^2 - \ddot{y}_1 \\ \dot{\mathbf{y}}_{\mathbf{1}}^3 - y_1 + 1 \end{array} \right.$$

— ∘ ◯ ∘ —

The following system corresponds to Euler equations for an incompressible fluid in two dimensions. The differential indeterminates v^1 and v^2 are the two components of the speed of the fluid. The differential indeterminate p is the pression. The three derivation operators are $\partial/\partial x_1$, $\partial/\partial x_2$ (the two space variables) and $\partial/\partial t$. Derivatives are denoted as follows: v_1^2 stands for $\partial/\partial x_1\, v^2$ and v_t^1 stands for $\partial/\partial t\, v^1$

$$A \left\{ \begin{array}{lcl} \mathbf{v_t^1} + v^1\, v_1^1 + v^2\, v_2^1 + p_1 & = & 0 \\ \mathbf{v_t^2} + v^1\, v_1^2 + v^2\, v_2^2 + p_2 & = & 0 \\ \mathbf{v_1^1} + v_2^2 & = & 0 \end{array} \right.$$

This system is not coherent because one Δ-polynomial is not reduced to zero by A (see details at the end of this document).

Question 36. Which Δ-polynomial?

Question 37. Is this system a regular differential chain anyway?

— ∘ ◯ ∘ —

The following system is a regular differential chain The differential indeterminates are u and v. The two derivation operators are $\partial/\partial x$ and $\partial/\partial y$

$$A \left\{ \begin{array}{lcl} \mathbf{v_{xx}} - u_x & = & 0 \\ 4\, u\, \mathbf{v_y} + u_x\, u_y - u_x\, u_y\, u & = & 0 \\ \mathbf{u_x}^2 - 4\, u & = & 0 \\ \mathbf{u_y}^2 - 2\, u & = & 0 \end{array} \right.$$

Question 38. Prove that $v_{xxx}\, u_{xx} - 4 \in [A] : H_A^\infty$ (see computation details at the end of the document)

Question 39. Prove[5] that $v_x - u \notin [A] : H_A^\infty$

[5] Hint: use the equivalence theorem for regular differential chains.

annihilating_polynomial

June 16, 2023

[2]:
```
from sympy import *
from DifferentialAlgebra import *
init_printing ()
```

[3]:
```
xi = var('xi')
x, y, z = indexedbase ('x, y, z')
```

[4]:
```
syst = [ Eq(y,x**3 + z), Eq(z**3,x**4), Eq(x[xi],1)]
syst
```

[4]: $\left[y = x^3 + z,\ z^3 = x^4,\ x_\xi = 1 \right]$

[5]:
```
R = DifferentialRing (derivations = [xi], blocks = [[y,z,x]], notation='jet')
```

[6]:
```
A = RegularDifferentialChain (syst, R)
```

[7]:
```
Rbar = DifferentialRing (derivations = [xi], blocks = [z,x,y], notation='jet')
```

[8]:
```
Abar = A.change_ranking (Rbar, prime=True)
```

[9]:
```
p = Abar.equations(solved=False)[0]
p
```

[9]: $19683 y_\xi^9 - 1594323 y_\xi^6 y^2 + 1456542 y_\xi^5 y - 253125 y_\xi^4 + 43046721 y_\xi^3 y^4 - 26572050 y_\xi^2 y^3 - 387420489 y^6 + 800000 y$

[10]:
```
ybar = xi**3 + xi**(4/Integer(3))
ybar
```

[10]: $\xi^{\frac{4}{3}} + \xi^3$

[11]:
```
simplify (Rbar.evaluate (p, {y:ybar}).doit())
```

[11]: 0

[ ]:
```

```

IO_equation_Lotka_Volterra

June 15, 2023

[1]:
```
from sympy import *
from DifferentialAlgebra import *
init_printing ()
```

[2]:
```
alpha, beta, mu, gamma = var ('alpha, beta, mu, gamma')
t = var('t')
y1, y2 = function ('y_1, y_2')
```

[3]:
```
params = [alpha, beta, mu, gamma]
R = DifferentialRing (
    derivations = [t],
    blocks = [[y1,y2], params],
    parameters = params)
```

[4]:
```
LotkaVolterraModel = [
    Eq(Derivative(y1(t),t), alpha*y1(t) - beta*y1(t)*y2(t)),
    Eq(Derivative(y2(t),t), mu*y1(t)*y2(t) - gamma*y2(t))]
LotkaVolterraModel
```

[4]:
$$\left[\frac{d}{dt}\mathrm{y}_1(t) = \alpha\,\mathrm{y}_1(t) - \beta\,\mathrm{y}_1(t)\,\mathrm{y}_2(t),\ \frac{d}{dt}\mathrm{y}_2(t) = -\gamma\,\mathrm{y}_2(t) + \mu\,\mathrm{y}_1(t)\,\mathrm{y}_2(t)\right]$$

[5]:
```
ideal = R.RosenfeldGroebner (LotkaVolterraModel)
ideal
```

[5]: [regular_differential_chain]

[6]:
```
IO_R = DifferentialRing (
    derivations = [t],
    blocks = [y2, y1, params],
    parameters = params)
```

[7]:
```
ideal = ideal[0]
IO_ideal = ideal.change_ranking (IO_R)
```

[8]:
```
IO_ideal.equations (solved=True)
```

[8]:

$$\left[\frac{d^2}{dt^2}\,\mathrm{y}_1(t) = \frac{\alpha\gamma\,\mathrm{y}_1{}^2(t) - \alpha\mu\,\mathrm{y}_1{}^3(t) - \gamma\,\mathrm{y}_1(t)\frac{d}{dt}\,\mathrm{y}_1(t) + \mu\,\mathrm{y}_1{}^2(t)\frac{d}{dt}\,\mathrm{y}_1(t) + \left(\frac{d}{dt}\,\mathrm{y}_1(t)\right)^2}{\mathrm{y}_1(t)},\ \mathrm{y}_2(t) = \frac{\alpha\,\mathrm{y}_1(t) - \frac{d}{dt}\,\mathrm{y}_1(t)}{\beta\,\mathrm{y}_1(t)}\right.$$

[9]:
```
IO_eq = IO_ideal.equations ()[0]
IO_eq = IO_eq / IO_R.initial (IO_eq)
IO_eq
```

[9]:

$$\frac{-\alpha\gamma\,\mathrm{y}_1{}^2(t) + \alpha\mu\,\mathrm{y}_1{}^3(t) + \gamma\,\mathrm{y}_1(t)\frac{d}{dt}\,\mathrm{y}_1(t) - \mu\,\mathrm{y}_1{}^2(t)\frac{d}{dt}\,\mathrm{y}_1(t) + \mathrm{y}_1(t)\frac{d^2}{dt^2}\,\mathrm{y}_1(t) - \left(\frac{d}{dt}\,\mathrm{y}_1(t)\right)^2}{\mathrm{y}_1(t)}$$

[10]:
```
IO_eq_integrated = IO_R.integrate (IO_eq, t)
IO_eq_integrated
```

[10]:

$$\left[\frac{-\alpha\gamma\,\mathrm{y}_1{}^2(t) + \alpha\mu\,\mathrm{y}_1{}^3(t) - \left(\frac{d}{dt}\,\mathrm{y}_1(t)\right)^2}{\mathrm{y}_1(t)},\ \gamma\,\mathrm{y}_1(t) - \frac{\mu\,\mathrm{y}_1{}^2(t)}{2},\ \mathrm{y}_1(t)\right]$$

[11]:
```
IO_eq_integrated = Add (*[Derivative (IO_eq_integrated[i], t, i) for i in range␣
↪(len(IO_eq_integrated))])
IO_eq_integrated
```

[11]:

$$\frac{-\alpha\gamma\,\mathrm{y}_1{}^2(t) + \alpha\mu\,\mathrm{y}_1{}^3(t) - \left(\frac{d}{dt}\,\mathrm{y}_1(t)\right)^2}{\mathrm{y}_1(t)} + \frac{\partial}{\partial t}\left(\gamma\,\mathrm{y}_1(t) - \frac{\mu\,\mathrm{y}_1{}^2(t)}{2}\right) + \frac{d^2}{dt^2}\,\mathrm{y}_1(t)$$

[ ]:
```
```

formal_power_series

June 15, 2023

[1]:
```
from sympy import *
from DifferentialAlgebra import *
init_printing ()
```

[2]:
```
x = var('x')
y = indexedbase('y')
```

[3]:
```
R = DifferentialRing (derivations = [x], blocks = [y], notation = 'jet')
```

[4]:
```
ybar = Add (*[y[i]*x**i/factorial(i) for i in range (0,5)])
ybar
```

[4]:
$$\frac{x^4 y_4}{24} + \frac{x^3 y_3}{6} + \frac{x^2 y_2}{2} + x y_1 + y_0$$

[5]:
```
p = y*y[x]**2 + y - 1
```

[6]:
```
valp = R.evaluate (p, {y:ybar}).doit ()
valp = collect (expand (valp), x)
valp
```

[6]:
$$\frac{x^{10} y_4^3}{864} + \frac{5x^9 y_3 y_4^2}{432} + x^8 \left(\frac{y_2 y_4^2}{36} + \frac{11 y_3^2 y_4}{288}\right) + x^7 \left(\frac{y_1 y_4^2}{24} + \frac{13 y_2 y_3 y_4}{72} + \frac{y_3^3}{24}\right) + x^6 \left(\frac{y_0 y_4^2}{36} + \frac{19 y_1 y_3 y_4}{72} + \frac{5 y_2^2 y_4}{24} + \frac{7 y_2 y_3^2}{24}\right) + x^5 \left(\frac{y_0 y_3 y_4}{6} + \frac{7 y_1 y_2 y_4}{12} + \frac{5 y_1 y_3^2}{12} + \frac{2 y_2^2 y_3}{3}\right) + x^4 \left(\frac{y_0 y_2 y_4}{3} + \frac{y_0 y_3^2}{4} + \frac{3 y_1^2 y_4}{8} + \frac{11 y_1 y_2 y_3}{6} + \frac{y_2^3}{2} + \frac{y_4}{24}\right) + x^3 \left(\frac{y_0 y_1 y_4}{3} + y_0 y_2 y_3 + \frac{7 y_1^2 y_3}{6} + 2 y_1 y_2^2 + \frac{y_3}{6}\right) + x^2 \left(y_0 y_1 y_3 + y_0 y_2^2 + \frac{5 y_1^2 y_2}{2} + \frac{y_2}{2}\right) + x \left(2 y_0 y_1 y_2 + y_1^3 + y_1\right) + y_0 y_1^2 + y_0 - 1$$

[7]:
```
valp = valp.subs ({y[0]:1,y[1]:2,y[2]:-2,y[3]:-1})
valp
```

[7]:
$$\frac{x^{10} y_4^3}{864} - \frac{5x^9 y_4^2}{432} + x^8 \left(-\frac{y_4^2}{18} + \frac{11 y_4}{288}\right) + x^7 \left(\frac{y_4^2}{12} + \frac{13 y_4}{36} - \frac{1}{24}\right) + x^6 \left(\frac{y_4^2}{36} + \frac{11 y_4}{36} - \frac{7}{12}\right) + x^5 \left(-\frac{5 y_4}{2} - \frac{11}{6}\right) + x^4 \left(\frac{7 y_4}{8} + \frac{43}{12}\right) + x^3 \left(\frac{2 y_4}{3} + \frac{79}{6}\right) - 19x^2 + 2x + 4$$

[8]:
```
renaming_dict = {y:y[0], y[x]:y[1], y[x,x]:y[2], y[x,x,x]:y[3], y[x,x,x,x]:y[4]}
p0 = R.evaluate (p, renaming_dict)
p0
```

[8]:
$$y_0 y_1^2 + y_0 - 1$$

[9]:
```
p1 = R.evaluate (R.differentiate (p, x), renaming_dict)
p1
```

[9]:
$$2y_0 y_1 y_2 + y_1^3 + y_1$$

[10]:
```
p2 = R.evaluate (R.differentiate (p, x**2), renaming_dict)
p2
```

[10]:
$$2y_0 y_1 y_3 + 2y_0 y_2^2 + 5y_1^2 y_2 + y_2$$

[11]:
```
p3 = R.evaluate (R.differentiate (p, x**3), renaming_dict)
p3
```

[11]:
$$2y_0 y_1 y_4 + 6y_0 y_2 y_3 + 7y_1^2 y_3 + 12y_1 y_2^2 + y_3$$

[12]:
```
[p.subs ({y[0]:1,y[1]:2,y[2]:-2,y[3]:-1})
     for p in [p0, p1, p2/factorial(2), p3/factorial(3)]]
```

[12]:
$$\left[4,\ 2,\ -19,\ \frac{2y_4}{3} + \frac{79}{6}\right]$$

[ ]:

ideal_decomposition

June 17, 2023

[1]:
```
from sympy import *
from sympy.polys.polytools import groebner
init_printing ()
```

[2]:
```
z, t1, x1, x2, x3 = var ('z, t1, x1, x2, x3')
```

[3]:
```
A = [t1*x3**2 - x2**2 - t1, x2**2 - t1**2, x1]
p3, p2, p1 = A
(p3, p2, p1)
```

[3]:

$$\left(t_1 x_3^2 - t_1 - x_2^2,\ -t_1^2 + x_2^2,\ x_1\right)$$

[4]:
```
s2 = diff(p2,x2)
s3 = diff(p3,x3)
s2, s3
```

[4]:

$$\left(2x_2,\ 2t_1 x_3\right)$$

Let us check that the regular chain is squarefree

[5]:
```
resultant (s2, p2, x2)
```

[5]:

$$-4t_1^2$$

[6]:
```
resultant (resultant (s3, p3, x3), p2, x2)
```

[6]:

$$16t_1^8 + 32t_1^7 + 16t_1^6$$

Euler_Equations

June 15, 2023

[1]:
```
from sympy import *
from DifferentialAlgebra import *
init_printing ()
```

[2]:
```
x1, x2, t = var ('x_1, x_2, t')
p, v1, v2 = indexedbase ('p, v1, v2')
```

[3]:
```
R = DifferentialRing (derivations = [t,x1,x2], blocks = [[v1,v2,p]], notation =
↪'jet')
```

[4]:
```
EulerEq = [ v1[t] + v1*v1[x1] + v2*v1[x2] + p[x1],
            v2[t] + v1*v2[x1] + v2*v2[x2] + p[x2],
            v1[x1] + v2[x2]]
R.equations (EulerEq, solved = True)
```

[4]: $\left[v_{1t} = -p_{x_1} - v_{1x_1} v_1 - v_{1x_2} v_2,\ v_{2t} = -p_{x_2} - v_{2x_1} v_1 - v_{2x_2} v_2,\ v_{1x_1} = -v_{2x_2} \right]$

[5]:
```
ideal = R.RosenfeldGroebner (EulerEq)
ideal
```

[5]: [regular_differential_chain]

[6]:
```
A = ideal[0]
A.equations (solved = True)
```

[6]: $\left[v_{2t} = -p_{x_2} - v_{2x_1} v_1 - v_{2x_2} v_2,\ v_{1x_1} = -v_{2x_2},\ v_{1t} = -p_{x_1} - v_{1x_2} v_2 + v_{2x_2} v_1,\ p_{x_1,x_1} = -p_{x_2,x_2} - 2 v_{1x_2} v_{2x_1} - 2 v_{2x}^2 \right.$

Ritt_reduction

June 17, 2023

[1]:
```
from sympy import *
from DifferentialAlgebra import *
init_printing ()
```

[2]:
```
x, y = var ('x, y')
u, v = indexedbase ('u, v')
```

[3]:
```
R = DifferentialRing (derivations = [x,y], blocks = [[v,u]], notation='jet')
```

[4]:
```
syst = [v[x,x] - u[x], u[x,y]*v[y] - u + 1, u[x]**2 - 4*u]
syst
```

[4]:

$$\left[-u_x + v_{x,x},\ u_{x,y}v_y - u + 1,\ u_x^2 - 4u\right]$$

[5]:
```
ideal = R.RosenfeldGroebner (syst)
ideal
```

[5]: [regular_differential_chain]

[6]:
```
A = ideal[0]
p1, p2, p3, p4 = A.equations ()
(p1, p2, p3, p4)
```

[6]:

$$\left(u_y^2 - 2u,\ u_x^2 - 4u,\ -u_x u_y u + u_x u_y + 4v_y u,\ -u_x + v_{x,x}\right)$$

[7]:
```
A.differential_prem (v[x,x,x]*u[x,x]-4)
```

[7]:

$$(2u_x,\ 0)$$

[8]:
```
A.differential_prem (v[x] - u)
```

[8]:

$$(1,\ v_x - u)$$

[ ]:
```

```