The exercises are not supposed to be difficult. They are mostly designed to incite you to seek the relevant information in the slides.

1 Lecture 1

Question 1. Using axioms of derivations, prove that

$$\left(\frac{a}{b}\right)' = \frac{\dot a\, b - a\, \dot b}{b^2}$$

> **Correction.** Since $(a/b)\, b = a$ we have $((a/b)\, b)' = a'$. By the axioms of derivations, $((a/b)\, b)' = (a/b)'\, b + (a/b)\, b'$. Solving with respect to $(a/b)'$ we get the sought formula.

A *ranking* is a total order on the infinite set ΘY of the derivatives which satisfies the two following axioms, for all derivatives $v, w \in \Theta Y$ and every derivation δ:

1. $v \leq \delta v$ and
2. $v < w \Rightarrow \delta v < \delta w$.

Question 2. How many rankings exist in the case of a single differential indeterminate and a single derivation?

> **Correction.** One: $y < \dot y < \ddot y < \cdots$

Question 3. Does there exist a relationship between rankings and the admissible orderings of the Gröbner bases theory?

> **Correction.** Assume we have two derivations δ_x and δ_t. Derivatives have the form $\delta_x^d\, \delta_t^e\, y$ for some nonnegative integers d, e and some differential indeterminate y. These derivatives can thus be represented as monomials $x^d\, t^e$. Any ranking on these derivatives corresponds to an admissible ordering on the corresponding monomials and conversely. Actually, applying a differential elimination process over a system of linear PDE, with constant coefficients, in one differential indeterminate is equivalent to computing a Gröbner basis with respect to the admissible ordering induced by the ranking. From this, we see the very high complexity of differential elimination, which contains, as subcases, the two main tools for polynomial system solving: Gröbner bases and regular chain decompositions.

Question 4. What is the leading derivative of p, its initial and its separant?

$$p = y_1^2\, \dot y_1^3 + 6 \dot y_1\, \ddot y_1^2$$

> **Correction.** leading derivative $\ddot y_1$ (whatever the ranking), initial $6\, \dot y_1$ and separant $12\, \dot y_1\, \ddot y_1$

Question 5. Differentiate p. What about the initial of $\dot{p}$?

Correction. The axioms of rankings imply that the leading derivative of $\dot{p}$ is the derivative of the leading derivative of p. The initial of any derivative of p is the separant of p (because any derivative of a differential polynomial has degree 1 in its leading derivative).

Question 6. Prove $\dot{y}^3 \in [y^2]$

Correction. The differential polynomial $\dot{y}^3$ is obtained by computing some sort of S-polynomial between the two first derivatives of p:

$$\begin{array}{rcl} (y^2)' & = & 2\,y\,\dot{y} \\ (y^2)'' & = & 2\,y\,\ddot{y} - 2\,\dot{y}^2 \\ \dot{y}^3 & = & \ddot{y}\,(y^2)' - \dot{y}\,(y^2)'' \end{array}$$

A differential polynomial can generate infinitely many S-polynomials with its infinitely many derivatives. One can define Gröbner bases of differential ideals but they are infinite in general. In the PDE case, the membership problem to differential ideals of the form $[\Sigma]$ is undecidable (Umirbaev, 2016). The ODE case is still open.

Question 7. What can be said about the perfect differential ideal $\{y^2\}$?

Correction. We have $\{y^2\} = \{y\} = [y]$. It is the ideal of all the differential polynomials with a zero constant term.

Question 8. Let Σ be a set of differential polynomials. True or false? $1 \in [\Sigma] \Leftrightarrow 1 \in \{\Sigma\}$

Correction. True! A power of 1 belongs to $[\Sigma]$ if and only if $1 \in [\Sigma]$

Question 9. A differential ideal $\mathfrak{A}$ is defined as an ideal such that $p \in \mathfrak{A} \Rightarrow \dot{p} \in \mathfrak{A}$. Is it obvious that $[\Sigma]$ is the ideal (in the non differential sense) generated by $\Theta\Sigma$ which contains the elements of Σ and all their derivatives up to any order?

Correction. It is actually not very difficult since it is sufficient to prove that $(\Theta\Sigma)$ is a differential ideal. Assume p is a linear combination of terms of the form $q\,\theta f$, where q is a differential polynomial, $\theta \in \Theta$ and $f \in \Sigma$. Then δp is a linear combination of terms of the form $q\,\delta\theta f + (\delta q)\,\theta f$ and is seen to belong to $(\Theta\Sigma)$.

Question 10. It can be proved that

$$\{\dot y^2 - 4\,y\} \;=\; \{\dot y^2 - 4\,y,\ \ddot y - 2\} \cap \{y\}$$

where the right hand side differential ideals are prime. Prove that none of the right hand side differential ideals is included in the other one. Can we deduce that the left hand side differential ideal is not prime?

> **Correction.** Since $\{y\}$ is the ideal of all the differential polynomials with a zero constant term we have $\ddot y - 2 \notin \{y\}$ hence $\{\dot y^2 - 4\,y,\ \ddot y - 2\} \not\subset \{y\}$. On the other hand, if we had the converse inclusion, the two differential ideals $\{\dot y^2 - 4\,y\}$ and $\{y\}$ would be equal and the differential equation $\dot y^2 - 4\,y = 0$ would have $y(x) = 0$ as unique solution — which is not the case. All these arguments imply that $y\,(\ddot y - 2) \in \{\dot y^2 - 4\,y\}$ while none of the factors do thus $\{\dot y^2 - 4\,y\}$ is not prime. More generally, an irredundant intersection of prime ideals is never prime but is always radical.

Question 11. We are looking for a differential polynomial p which is annihilated by the following expression (computations at the end of the document):

$$\text{expr} \;=\; x^3 + x^{\frac{4}{3}}$$

Set up a differential elimination problem which permits to compute p. Which system? Which ranking?

> **Correction.** First rename the independent variable to ξ and view x as a differential indeterminate. The differential system is then (the last equation is necessary because x is no more the independent variable):
>
> $$\begin{aligned} y &= x^3 + z \\ z^3 &= x^4 \\ \dot x &= 1 \end{aligned}$$
>
> Since we are looking for a differential polynomial in y alone, the ranking must eliminate all other differential indeterminates: $(x, z) \gg y$. Observe that the differential elimination can be costly because of the case splitting mechanism. Over such an example, the best strategy consists in observing that it is already a regular differential chain with respect to the ranking $y \gg z \gg x$. The known regular differential chain permits to decide membership and regularity to the differential ideal that it defines hence to avoid splittings while performing the differential elimination process with respect to the desired ranking. This strategy is carried out by the `change_ranking` function of the software.

— ∘ ◯ ∘ —

Lotka-Volterra prey-predator model is

$$\begin{aligned}\dot y_1 &= \alpha\, y_1 - \beta\, y_1\, y_2\\ \dot y_2 &= \mu\, y_1\, y_2 - \gamma\, y_2\end{aligned}$$

The differential indeterminates y_1 and y_2 represent the densities of populations of two animal species. Greek letters are parameters.

A differential polynomial has been obtained by differential elimination over the model equations (see details at the end of this document). It has been divided by its initial. The result is

$$\frac{\mathsf{d}}{\mathsf{d}t}\left(\gamma\, y_1 - \frac{\mu\, y_1^2}{2}\right) + \alpha\,\mu\, y_1^2 - \alpha\,\gamma\, y_1 = \frac{\dot y_1^2}{y_1} - \ddot y_1$$

Question 12. Which ranking should be provided to the differential elimination software to obtain such an equation?

Correction. A ranking which eliminates y_2 i.e. $y_2 \gg (y_1, \alpha, \beta, \mu, \gamma)$.

Assume the knowledge of a function $y_1(t)$ satisfying the equation, so that any differential fraction in $\mathscr{F}{<}y_1{>}$ can be evaluated to numbers, at (say) $t = t_1, \ldots, t_9$.

Question 13. Write an overdetermined linear system $A\,x = b$, where A is matrix and b a vector of numbers, stating the equation is satisfied at $t_1, \ldots, t_9$. What are the dimension of A? the parameter blocks in x?

Correction. There are 9 equations (one for each t_i). The unknowns are the blocks of parameters $\gamma, \mu, \alpha\,\mu, \alpha\,\gamma$. Thus $\dim A = 9 \times 4$. The entries of A are numbers since they can be evaluated thanks to our knowledge of $y_1(t)$. The entries of b are the 9 numbers obtained by evaluating the right hand side of the differential equation at $t_1, \ldots, t_9$.

Question 14. Assume A has full rank so that $A\,x = b$ can be solved by linear least squares. Is it possible to recover the values of the model parameters?

Correction. The vector x which minimizes the error $\|A\,x - b\|_2$ is the solution of $A^T A\,x = A^T b$. Solving it, we get an approximation of γ and μ. With the approximation of (say) γ and $\alpha\,\gamma$ we get an approximation of α. Unfortunately, the parameter β cannot be obtained (it only occurs in the differential equation which depends on y_2).

2 Lecture 2

The differential $\mathscr{F}$-algebra of differential power series in one differential indeterminate y is denoted

$$\mathscr{F}\{\{y\}\}$$

An element of this algebra is a possibly infinite sum of terms of the form

$$c_i\, t_i$$

where the $c_i \in \mathscr{F}$ and the t_i are power products of y and its derivatives up to any order

Question 15. Draw a picture of Differential Power Serious.

Question 16. Compute the partial remainder of $\ddot{y}$ with respect to

$$p \;=\; \dot{y}^2 - 4\,y$$

How to relate this computation to the computation of formal power series solution of p?

> **Correction.** We have $s_p\,\ddot{y} = 4\,\dot{y} \mod [p]$. The remainder is $4\,\dot{y}$. The separant is $2\,\dot{y}$. When computing a formal power series solution of p, the coefficient y_2 is given by $4\,y_1/2\,y_1 = 2$.

— ∘ ◯ ∘ —

Question 17. Compute[1] p_0, p_1, p_2 defined by $p_i = p^{(i)}(y_0, y_1, y_2, y_3)$

$$p \;=\; y\,\dot{y}^2 + y - 1$$

> **Correction.** One finds
>
> $$\begin{aligned} p_0 &= y_0\,y_1^2 + y_0 - 1 \\ p_1 &= 2\,y_0\,y_1\,y_2 + y_1^3 + y_1 \\ p_2 &= 2\,y_0\,y_1\,y_3 + 2\,y_0\,y_2^2 + 5\,y_1^2\,y_2 + y_2 \end{aligned}$$

Question 18. Check that, for $y_0, y_1, y_2, y_3 = 1, 2, -2, -1$ we have $p_0, p_1, p_2 = 4, 2, -38$

Question 19. Use the information to compute $p(\bar{\bar{y}}) \mod x^3$ where

$$\bar{\bar{y}} \;=\; y_0 + y_1\,x + y_2\,\frac{x^2}{2} + \cdots$$

Check

> **Correction.** We have $p(\bar{\bar{y}}) = p_0 + p_1\,x + p_2\,\frac{x^2}{2} \mod x^3$ hence, using the above values, $p(\bar{\bar{y}}) = 4 + 2\,x - 19\,x^2 \mod x^3$.

[1] Computation details at the end of this document.

Question 20. Assume p is a linear differential equation in one differential indeterminate and constant coefficients. Does the general method provided during the lecture simplify?

Question 21. Provide an example of a triangular system of ordinary differential polynomials the elements of which are not pairwise partially reduced.

Provide an example of a triangular system of partial differential polynomials which is not coherent.

In each case, provide a ranking.

> **Correction.** In the first case, consider $\dot{y}_2 + \ddot{y}_1, \dot{y}_1$ for the ranking $y_2 \gg y_1$ so that $\dot{y}_2$ is the leading derivative of the first differential polynomial. In the second case, consider $u_x + v, u_y$ for any ranking such that $u \gg v$ so that u_x is the leading derivative of the first differential polynomial.

Question 22. Provide an example of a triangular system A of differential polynomials, the elements of which are pairwise partially reduced, which is coherent but such that the following system has no solution[2]

$$A \;=\; 0\,, \qquad H_A \;\neq\; 0\,.$$

> **Correction.** The common factors of a polynomial and its separant are the multiple factors of the polynomial. Thus if an element of A has only multiple factors, the system has no solution. an answer could be $u_x + v, u_y, v^2$ which is coherent since the Δ-polynomial v_y is reduced to zero by v^2.

— ∘ ◯ ∘ —

Let Σ be a differential polynomial system

Question 23. Among the two following problems, which one is equivalent to the problem of deciding if $1 \in \{\Sigma\}$?

1. The existence problem of formal power series solutions for prescribed initial values

2. The existence problem of formal power series solutions for some, unspecified, initial values

> **Correction.** Problem **2.**

[2]Hint: what if some element of A has multiple factors?

Question 24. Is the problem of deciding if $1 \in \{\Sigma\}$ algorithmic?

> **Correction.** Yes! $1 \in \{\Sigma\}$ if and only if the differential elimination method outputs an empty list of regular differential chains.

— ∘ ◯ ∘ —

Hilbert's tenth problem: provide a general algorithm which, for any given Diophantine equation (a polynomial equation with integer coefficients and a finite number of unknowns), can decide whether the equation has a solution with all unknowns taking integer values.

In 1970, Yuri Matiyasevich proved that no such algorithm exists whenever the number of unknowns is greater than or equal to 9.

Question 25. In which cases the following system has a nonzero formal power series solution?

$$\begin{aligned} x\,\dot{y} &= \alpha\,y \\ \dot{\alpha} &= 0 \end{aligned}$$

> **Correction.** The system admits $y(x) = 0$ as a solution. Moeover, α is a non-negative integer if and only if the formal power series x^{α} also is a solution.

Question 26. Prove that there does not exist any algorithm which decides if a given arbitrary system of ordinary differential polynomials has a nonzero formal power series solution.

> **Correction.** This is an undecidable problem. Make a system with 9 different copies of the above one and add $P(\alpha_1, \ldots, \alpha_9) = 0$ where P is a polynomial in 9 variables and integer coefficients. Then this big system has a nonzero formal power series solution if and only if $P(\alpha_1, \ldots, \alpha_9) = 0$ has nonnegative integer solutions. Therefore, if we could decide whether any given differential polynomial system had nonzero formal power series solutions, we could solve Hilbert's tenth problem: a contradiction to Matiyasevich theorem.

3 Lecture 3

Question 27. Prove that $x_2 \in (A) : I_A^{\infty}$ where $A = x_1\,x_2$

> **Correction.** We have $I_A = x_1$ hence $(A) : I_A^{\infty} = (x_1)$.

Question 28. True of false? A triangular set A is a basis of the ideal $(A) : I_A^{\infty}$. If true, why; if false, provide a counter-example.

> **Correction.** False! See above.

Question 29. Give a technique which permits to compute a basis of $(A) : I_A^\infty$.

> **Correction.** Compute a Gröbner basis B of $(A,\, I_A\, z - 1)$ where z is a new variable, for any admissible ordering such that $z \gg (x_i, t_j)$. A Gröbner basis of (A) is given by the elements of B which do not depend on z.

Question 30. Recall the definition of a zerodivisor modulo an ideal $\mathfrak{A}$. Prove that the initials of A are regular (= non zerodivisors) modulo $(A) : I_A^\infty$.

> **Correction.** Indeed, h is a zerodivisor modulo $\mathfrak{A}$ if there exists some $f \notin \mathfrak{A}$ such that $h\, f \in \mathfrak{A}$. Now, if h is a power product of initials of A then we have $h\, f \in (A) : I_A^\infty \Rightarrow f \in (A) : I_A^\infty$. Thus h is regular modulo $(A) : I_A^\infty$.

Question 31. Is the following system a regular chain?

$$A \left\{ \begin{array}{l} (x_1^2 - t_1^2)\, \mathbf{x_2} - x_1 + t_2 \\ \mathbf{x_1}\, (\mathbf{x_1} - t_1) \end{array} \right.$$

> **Correction.** No: the initial of p_2 has a common factor with p_1 hence is not regular modulo (p_1).

Question 32. Is[3] the following system a regular chain?

$$A \left\{ \begin{array}{l} (t_1^2 - t_1)\, \mathbf{x_2} - x_1 + t_2 \\ \mathbf{x_1}\, (\mathbf{x_1} - t_1) \end{array} \right.$$

> **Correction.** Yes: we only need to check that the initial of p_2 is regular modulo (p_1). For this, it is sufficient to check that the resultant of this initial and p_1 with respect to x_1 is nonzero. Since the initial does not depend on x_1, the resultant is a power of it and is thus nonzero.

Question 33. Is the following system a squarefree regular chain?

$$A \left\{ \begin{array}{l} \mathbf{x_2}^2 + x_1^2 - t_1 \\ \mathbf{x_1}^2 - t_1 \end{array} \right.$$

> **Correction.** No: modulo (p_1), polynomial p_2 has a multiple factor thus its separant is a zerodivisor modulo (p_1) and A is not squarefree.

[3] Hint: use the equivalence theorem for regular chains

Question 34. Consider the following system

$$A \left\{ \begin{array}{l} t_1\,\mathbf{x_3}^2 - t_1 - x_2^2 \\ \mathbf{x_2}^2 - t_1^2 \\ x_1 \end{array} \right.$$

Compute a decomposition of the radical of the ideal (A) as an intersection of ideals defined by regular chains by applying the following strategy (computations detailed at the end of this document):

1. By splitting cases decompose the solution set of $A = 0$ into systems of the form $A_i = 0,\ I_{A_i} \neq 0$ where the A_i are regular chains.
2. Using the Theorem of Zeros (Nullstellensatz), transform this decomposition as a representation of the radical of the ideal (A) as an intersection of the radicals of the ideals defined by the regular chains A_i.
3. Prove that the regular chains A_i are squarefree. How does this information simplify the above formula?

   Correction. The only non constant initial is t_1 (thus $I_A = t_1$). Thus the solution set of $A = 0$ can be decomposed as the union of the solution sets of $A = 0,\ I_A \neq 0$ and $A = 0,\ t_1 = 0$. This latter system simplifies to $x_1 = t_1 = x_2 = 0$. By the Nullstellensatz, we thus have

   $$\sqrt{(A)} \quad = \quad \sqrt{(A) : I_A^\infty} \cap (t_1, x_1, x_2)\,.$$

   The resultants of the separants s_2 and s_3 with respect to A are non zero. Thus A is squarefree, $(A) : I_A^\infty$ is radical and

   $$\sqrt{(A)} \quad = \quad (A) : I_A^\infty \cap (t_1, x_1, x_2)\,.$$

   In general, when applying a regular chain decomposition algorithm on a polynomial system A, the intersection of the ideals defined by the regular chains is not necessarily equal to (A) but to some ideal $\mathfrak{A}$ such that $(A) \subset \mathfrak{A} \subset \sqrt{(A)}$.

Question 35. Is the following system a regular differential chain?

$$A \left\{ \begin{array}{l} \dot{\mathbf{y}}_{\mathbf{2}}^2 - \ddot{y}_1 \\ \dot{\mathbf{y}}_{\mathbf{1}}^3 - y_1 + 1 \end{array} \right.$$

Correction. No: p_2 is not partially reduced with respect to p_1.

— ∘ ◯ ∘ —

The following system corresponds to Euler equations for an incompressible fluid in two dimensions. The differential indeterminates v^1 and v^2 are the two components of the speed of the fluid. The differential indeterminate p is the pression. The three derivation operators are $\partial/\partial x_1$, $\partial/\partial x_2$ (the two space variables) and $\partial/\partial t$. Derivatives are denoted as follows: v^2_1 stands for $\partial/\partial x_1\, v^2$ and v^1_t stands for $\partial/\partial t\, v^1$

$$A \left\{ \begin{array}{lcl} \mathbf{v^1_t} + v^1\, v^1_1 + v^2\, v^1_2 + p_1 & = & 0 \\ \mathbf{v^2_t} + v^1\, v^2_1 + v^2\, v^2_2 + p_2 & = & 0 \\ \mathbf{v^1_1} + v^2_2 & = & 0 \end{array} \right.$$

This system is not coherent because one Δ-polynomial is not reduced to zero by A (see details at the end of this document).

Question 36. Which Δ-polynomial?

> **Correction.** The only possible Δ-polynomial is the one defined by p_1 and p_3. It is equal to $\delta_t\, p_1 - \delta_1\, p_3$.

Question 37. Is this system a regular differential chain anyway?

> **Correction.** No: if a Δ-polynomial is not reduced to zero then A is not coherent (note: strictly speaking, this is not exactly true) and cannot be a regular differential chain. See computations at the end of this document to see the regular differential chain obtained after a few steps. Note that the ranking has been chosen carefully: when printed in a text file, regular differential chains for rankings such that $(p, v^2) \gg v^1$ have sizes between half a megabyte and one megabyte. Existing algorithms have not been able to compute regular differential chains for this system and rankings such that $(v^1, v^2) \gg p$.

— ∘ ◯ ∘ —

The following system is a regular differential chain The differential indeterminates are u and v. The two derivation operators are $\partial/\partial x$ and $\partial/\partial y$

$$A \left\{ \begin{array}{lcl} \mathbf{v_{xx}} - u_x & = & 0 \\ 4\,u\,\mathbf{v_y} + u_x\, u_y - u_x\, u_y\, u & = & 0 \\ \mathbf{u_x}^2 - 4\,u & = & 0 \\ \mathbf{u_y}^2 - 2\,u & = & 0 \end{array} \right.$$

Question 38. Prove that $v_{xxx}\, u_{xx} - 4 \in [A] : H_A^\infty$ (see computation details at the end of the document)

Correction. The differential polynomial is reduced to zero by A using Ritt's full reduction method. Note that this regular differential chain has the property that its formal power series solutions depend on only three arbitrary constants (corresponding to v_x, v and u). Moreover, all its solutions are polynomials. This can be proved by testing that, above order 3, all the derivatives of u and v are reduced to zero by A.

Question 39. Prove[4] that $v_x - u \notin [A] : H_A^\infty$

Correction. This differential polynomial is equal to its own remainder by A using Ritt's full reduction method. Thus it does not belong to $[A] : H_A^\infty$

[4]Hint: use the equivalence theorem for regular differential chains.