Formal Power Series and Differential Elimination

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Formal Power Serious

Picture from getdrawings.com

Ritt's Reduction Algorithm

It is a generalization of pseudo-division

Its efficiency relies on the fact that the derivative of a differential polynomial has degree one in its leading derivative

$$(v^n)' = n\, v^{n-1}\, \dot{v}$$

This would be false in difference algebra

If $p, f \in \mathscr{R}[x]$ are two polynomials

$$f \quad = \quad f_e\, x^e + \cdots + f_1\, x + f_0 \qquad p \quad = \quad p_d\, x^d + \cdots + p_1\, x + p_0$$

Then there exist $q, g \in \mathscr{R}[x]$ with such that

$$p_d^{e-d+1}\, f \quad = \quad q\, p + g \quad (\deg(g, x) < d)$$

The polynomial $g = \mathrm{prem}(f, p, x)$ is the pseudoremainder of f by p

Assume f, p are differential polynomials and a ranking is fixed

$$p = \dot{y}^2 + 8\,x\,y - y$$

It is possible to compute the pseudoremainder of f by $p, \dot{p}, \ddot{p}, \ldots$
This process is called Ritt's reduction

The leading derivative $\dot{y}$ of p plays the role of x
The initial i_p $(= 1)$ of p plays the role of p_d when reducing by p
The separant s_p $(= 2\,\dot{y})$ of p, which is the initial of every derivative of p, plays the role of p_d when reducing by any derivative of p

Specifications of Ritt's Reduction (v1)

There exists $d, e \in \mathbb{N}$ and $g, q_0, q_1, \cdots \in \mathscr{F}\{y\}$ such that

$$i_p^d\, s_p^e\, f = g + q_0\, p + q_1\, \dot{p} + \cdots$$

Assume f, p are differential polynomials and a ranking is fixed

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Specifications of Ritt's Reduction (v2)

There exists a power product h of the initial and the separant of p and $g \in \mathscr{F}\{y\}$ such that

$$h\,f = g \mod [p]$$

Assume f, p are differential polynomials and a ranking is fixed

$$p = \dot{y}^2 + 8\,x\,y - y$$

It is possible to compute the pseudoremainder of f by $p, \dot{p}, \ddot{p}, \ldots$
This process is called Ritt's reduction

There exists a power product h of i_p, s_p and $g \in \mathscr{F}\{y\}$ such that

$$h\,f = g \mod [p]$$

Assume f, p are differential polynomials and a ranking is fixed

$$p = \dot{y}^2 + 8\,x\,y - y$$

It is possible to compute the pseudoremainder of f by $p, \dot{p}, \ddot{p}, \ldots$
This process is called Ritt's reduction

There exists a power product h of i_p, s_p and $g \in \mathscr{F}\{y\}$ such that

$$h\,f = g \mod [p]$$

The process can be a **full** or a **partial** reduction

partial: g is free of $\ddot{y}, y^{(3)}, \ldots$ (proper derivatives of $\dot{y}$)
full: in addition, $\deg(g, \dot{y}) < \deg(p, \dot{y})$

Remark The property of the partial remainder relies on the fact that proper derivatives of p have degree 1 in their leading derivatives

$$\dot{p} = 2\,\dot{y}\,\ddot{y} + 8\,x\,\dot{y} - \dot{y} + 8\,y$$

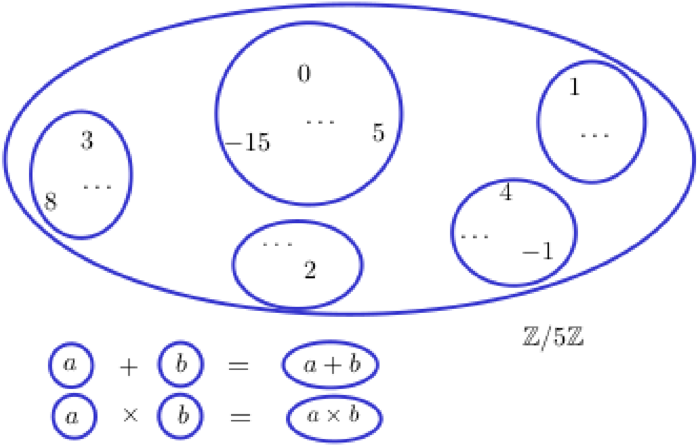
Solving in Abstract Differential Fields

It is the spirit of Ritt's *Differential Algebra* or Kolchin's *Differential Algebra and Algebraic Groups*

Though terribly abstract, it is sometimes the simplest and the most general point of view to have

To be combined with Seidenberg's *Embedding Theorem* which states that every abstract differential field is isomorphic to a field of formal power series

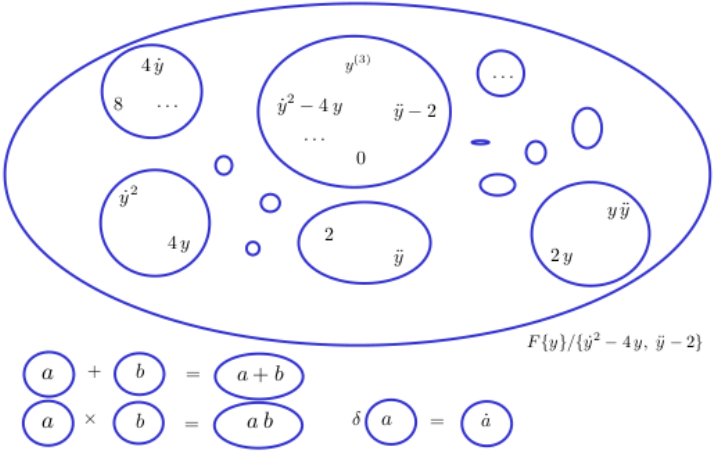
A. Seidenberg. *Differential Algebra and the Analytic Case*. 1958



$\mathbb{Z}/5\mathbb{Z}$ is a set of five equivalence classes

It is endowed with a ring structure by defining the sum and the product of two classes

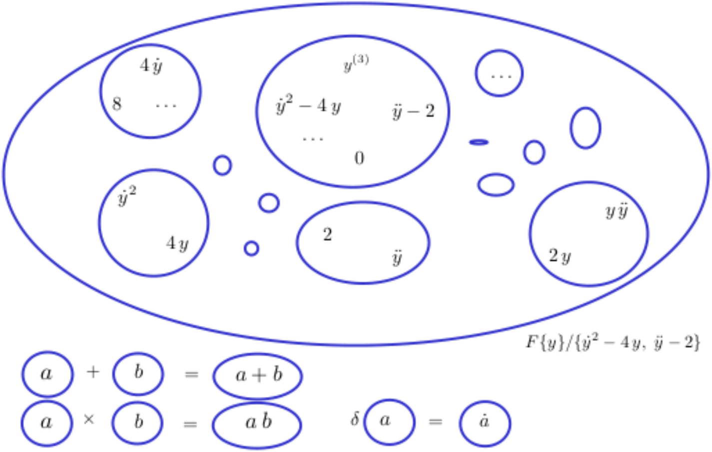
The definitions make sense because the class of $a + b$ (resp. $a \times b$) depends on the classes of a and b, not of a and b



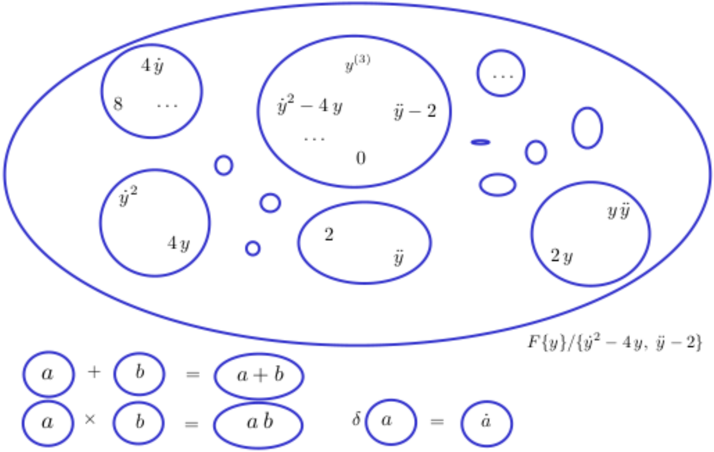
$\mathfrak{P} = \{\dot{y}^2 - 4\,y,\ \ddot{y} - 2\}$
$\mathscr{F}\{y_1, \ldots, y_n\}/\mathfrak{P}$ is a set of infinitely many equivalence classes

It can be endowed with a differential ring structure

By considering pairs of equivalence classes, one can define the differential field of fractions of $\mathscr{F}\{y\}/\mathfrak{P}$ provided that $\mathfrak{P}$ is prime



The solution in $\mathscr{F}\{y\}/\mathfrak{P}$ of $\mathfrak{P} = \{\dot{y}^2 - 4\,y\,,\ddot{y} - 2\}$ is . . .



The solution in $\mathscr{F}\{y\}/\mathfrak{P}$ of $\mathfrak{P} = \{\dot{y}^2 - 4\,y\,, \ddot{y} - 2\}$ is …

By the *Embedding Theorem* [Seidenberg, 1958], every such abstract differential field is isomorphic to a field of formal power series

Reduction to the Autonomous Case Almost Always Possible

A differential polynomial p in **non autonomous** if p explicitly depends on x

$$x\,\dot{y} + 8\,(x-2)\,y^2 - 1 \;=\; 0 \tag{1}$$

In Ritt and Kolchin differential algebra, there is no such thing as a non autonomous differential polynomial

At the price of one extra differential indeterminate and one extra differential equation (per derivation), the reduction process is always possible provided that coefficients belong to $\mathscr{F}[x]$ or $\mathscr{F}(x)$

$$\begin{aligned} x\,\dot{y} + 8\,(x-2)\,y^2 - 1 &= 0 \\ \dot{x} - 1 &= 0 \end{aligned} \tag{2}$$

Eq (1) has no formal power series solution centered at $x = 0$

Syst (2) has no formal power series solution for i.v. $x(0) = x_0 = 0$

Summary The reduction process transforms issues on expansion points into issues on initial values

Solving in Formal Power Series: Principle and Easy Case

Many tropical differential geometry problems consider non autonomous equations with coefficients in $\mathscr{F}[[x]]$

Such settings address the existence problem of formal power series solutions centered at the origin

In the easy case, we address a simpler but related problem: does there exist initial values for which formal power series solutions exist?

Recall the Brachistochrone equation

$$y\,\dot{y}^2 + y \quad = \quad D \quad (D \text{ nonzero constant})$$

It has formal power series solution but not for $y_0 = 0$

The Renaming Step

A differential equation states an equality between functions for every value of the independent variable

$$\dot{y}(x)^2 \;=\; 4\,y(x) \quad (\forall\, x)$$

The "renaming step" (next slide) expresses the fact that, the equality holds in particular for at $x = 0$ i.e. for "initial values" $y_0, y_1, \dots = y(0), \dot{y}(0), \dots$

$$y_1^2 \;=\; 4\,y_0$$

It is sometimes convenient to say we look for a "non differential solution" of

$$\dot{y}^2 \;=\; 4\,y$$

Looking for a formal power series centered at $x = \alpha$

$$\bar{\bar{y}} = y_0 + y_1 (x-\alpha) + y_2 \frac{(x-\alpha)^2}{2} + y_3 \frac{(x-\alpha)^3}{6} + \cdots$$

solution of

$$p(x,y) = \dot{y}^2 + 8\,x\,y - y$$

Step 1: differentiate p

$$\begin{aligned} \dot{y}^2 + 8\,x\,y - y &= 0 \\ 2\,\dot{y}\,\ddot{y} + 8\,x\,\dot{y} - \dot{y} + 8\,y &= 0 \\ 2\,\dot{y}\,y^{(3)} + 2\,\ddot{y}^2 + 8\,x\,\ddot{y} - \ddot{y} + 16\,\dot{y} &= 0 \end{aligned}$$

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Step 2: rename $y^{(i)}$ as y_i

$$\begin{aligned} y_1^2 + 8\,x\,y_0 - y_0 &= 0 \\ 2\,y_1\,y_2 + 8\,x\,y_1 - y_1 + 8\,y_0 &= 0 \\ 2\,y_1\,y_3 + 2\,y_2^2 + 8\,x\,y_2 - y_2 + 16\,y_1 &= 0 \end{aligned}$$

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Step 2: rename $y^{(i)}$ as y_i

$$\begin{aligned} y_1^2 + 8\,x\,y_0 - y_0 &= 0\\ 2\,y_1\,y_2 + 8\,x\,y_1 - y_1 + 8\,y_0 &= 0\\ 2\,y_1\,y_3 + 2\,y_2^2 + 8\,x\,y_2 - y_2 + 16\,y_1 &= 0 \end{aligned}$$

Step 3: evaluate at $x = \alpha$ and denote $p_i = p^{(i)}(\alpha, y_0, y_1, y_2, \ldots)$

$$\begin{array}{lll} p_0 & y_1^2 + 8\,\alpha\,y_0 - y_0 & = \; 0\\ p_1 & 2\,y_1\,y_2 + 8\,\alpha\,y_1 - y_1 + 8\,y_0 & = \; 0\\ p_2 & 2\,y_1\,y_3 + 2\,y_2^2 + 8\,\alpha\,y_2 - y_2 + 16\,y_1 & = \; 0 \end{array}$$

Looking for a formal power series centered at $x = \alpha$

$$\bar{\bar{y}} \;=\; y_0 + y_1\,(x-\alpha) + y_2\,\frac{(x-\alpha)^2}{2} + y_3\,\frac{(x-\alpha)^3}{6} + \cdots$$

solution of

$$p(x,y) \;=\; \dot{y}^2 + 8\,x\,y - y$$

Step 1: differentiate p
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$$\begin{array}{lll} p_0 & y_1^2 + 8\,\alpha\,y_0 - y_0 & = \; 0 \\ p_1 & 2\,y_1\,y_2 + 8\,\alpha\,y_1 - y_1 + 8\,y_0 & = \; 0 \\ p_2 & 2\,y_1\,y_3 + 2\,y_2^2 + 8\,\alpha\,y_2 - y_2 + 16\,y_1 & = \; 0 \end{array}$$

Looking for a formal power series centered at $x = \alpha$

$$\bar{\bar{y}} = y_0 + y_1\,(x-\alpha) + y_2\,\frac{(x-\alpha)^2}{2} + y_3\,\frac{(x-\alpha)^3}{6} + \cdots$$

solution of

$$p(x, y) = \dot{y}^2 + 8\,x\,y - y$$

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Fact

$$p(x, \bar{\bar{y}}) = p_0 + p_1\,(x-\alpha) + p_2\,\frac{(x-\alpha)^2}{2} + p_3\,\frac{(x-\alpha)^3}{6} + \cdots$$

Looking for a formal power series centered at $x = \alpha$

$$\bar{\bar{y}} \quad = \quad y_0 + y_1\,(x-\alpha) + y_2\,\frac{(x-\alpha)^2}{2} + y_3\,\frac{(x-\alpha)^3}{6} + \cdots$$

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$$p(x,y) \quad = \quad \dot{y}^2 + 8\,x\,y - y$$

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$$\begin{array}{lll} p_0 & y_1^2 + 8\,\alpha\,y_0 - y_0 & = \quad 0 \\ p_1 & 2\,y_1\,y_2 + 8\,\alpha\,y_1 - y_1 + 8\,y_0 & = \quad 0 \\ p_2 & 2\,y_1\,y_3 + 2\,y_2^2 + 8\,\alpha\,y_2 - y_2 + 16\,y_1 & = \quad 0 \end{array}$$

Notice the separant $s_p = 2\,\dot{y}$, the initial of every derivative of p, which provides the leading coefficient of $p_1, p_2, \ldots$

$$\bar{\bar{y}} = y_0 + y_1 (x-\alpha) + y_2 \frac{(x-\alpha)^2}{2} + y_3 \frac{(x-\alpha)^3}{6} + \cdots \quad (3)$$

$$p(x,y) = \dot{y}^2 + 8\,x\,y - y \quad (4)$$

Steps 1, 2, 3: differentiate, rename, evaluate at $x = \alpha$

$$\begin{array}{lll} p_0 & y_1^2 + 8\,\alpha\,y_0 - y_0 & = 0 \\ p_1 & 2\,y_1\,y_2 + 8\,\alpha\,y_1 - y_1 + 8\,y_0 & = 0 \end{array}$$

Step 4: solve and substitute the solution in $\bar{\bar{y}}$

$$\bar{\bar{y}} = y_0 + y_1(x-\alpha) + y_2 \frac{(x-\alpha)^2}{2} + y_3 \frac{(x-\alpha)^3}{6} + \cdots \quad (3)$$

$$p(x,y) = \dot{y}^2 + 8\,x\,y - y \quad (4)$$

Steps 1, 2, 3: differentiate, rename, evaluate at $x = \alpha$

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Step 4: solve and substitute the solution in $\bar{\bar{y}}$

Example Looking for y_2? Compute the partial remainder g of $\ddot{y}$ by p

$$\dot{p} = 2\,\dot{y}\,\ddot{y} + 8\,x\,\dot{y} - \dot{y} + 8\,y$$

Thus

$$\underbrace{(2\,\dot{y})}_{h}\,\ddot{y} = \underbrace{-8\,x\,\dot{y} + \dot{y} - 8\,y}_{g} \mod [p]$$

and

$$y_2 = \frac{g(\alpha, y_0, y_1)}{h(\alpha, y_0, y_1)}$$

$$\bar{\bar{y}} = y_0 + y_1 (x - \alpha) + y_2 \frac{(x-\alpha)^2}{2} + y_3 \frac{(x-\alpha)^3}{6} + \cdots \quad (3)$$

$$p(x, y) = \dot{y}^2 + 8\, x\, y - y \quad (4)$$

Steps 1, 2, 3: differentiate, rename, evaluate at $x = \alpha$

$$\begin{array}{lll} p_0 & y_1^2 + 8\,\alpha\, y_0 - y_0 & = 0 \\ p_1 & 2\, y_1\, y_2 + 8\,\alpha\, y_1 - y_1 + 8\, y_0 & = 0 \end{array}$$

Step 4: solve and substitute the solution in $\bar{\bar{y}}$

Summary Every solution of the following system on initial values can be prolongated to a differential solution $\bar{\bar{y}}$

$$y_1^2 + 8\,\alpha\, y_0 - y_0 = 0\,, \quad 2\, y_1 \neq 0\,.$$

Reformulation Every "non differential" zero of p which does not annihilate the initial and separant of p can be prolongated into a differential zero

A Famous Ritt Example

The solution set of

$$\dot{y}^2 - 4\,y \;=\; 0$$

can be decomposed in two cases

$$\underbrace{\left\{\begin{array}{rcl} \dot{y}^2 & = & 4\,y\,, \\ 2\,\dot{y} & \neq & 0 \end{array}\right.}_{y(x)=(x+c)^2} \quad \text{and} \quad \underbrace{y \;=\; 0}_{y(x)=0}$$

The general solution $y(x) = (x+c)^2$ corresponds to initial values

$$y_0,\; y_1,\; y_2 \;=\; c^2,\; 2\,c,\; 2$$

One of these formal power series solutions ($y(x) = x^2$) satisfies

$$2\,\dot{y} \;\neq\; 0 \qquad 2\,y_1 \;=\; 0\,.$$

The study of these formal power series solutions belong to the difficult case

Generalization of the Easy Case to Systems

$\mathscr{F}\{y_1, \ldots, y_n\}$

$A = \{p_1, \ldots, p_r\}$ a triangular set of differential polynomials

ODE case: the elements of A must be pairwise partially reduced

PDE case: in addition, A must be coheremt

Note The coherence condition is due to A. Rosenfeld (1959) who generalizes a result of A. Seidenberg. Kolchin's definition hides its algorithmic nature

A. Rosenfeld. *Specializations in Differential Algebra*. 1959

A. Seidenberg *An Elimination Theory in Differential Algebra*. 1956

Remark Up to some encoding, Gröbner bases (1965) are a particular case of coherent systems

The Pairwise Partially Reduced Condition

$\mathscr{F}\{y, z\}$ with a single derivation

$A = \{p_1, \ldots, p_r\}$ a triangular set of differential polynomials

$H_A = i_1 \cdots i_r \, s_1 \cdots s_r$ the product of the initials and separants of A

The elements of$A$ are **not pairwise partially reduced**

$$A \left\{ \begin{array}{rcl} z & = & \ddot{y}\,, \\ \dot{y}^2 & = & 4\,y\,. \end{array} \right.$$

At steps 3 and 4, the constraint $y_2 = 2$ is missed (assuming $y_1 \neq 0$)

The Pairwise Partially Reduced Condition

$\mathscr{F}\{y, z\}$ with a single derivation

$A = \{p_1, \ldots, p_r\}$ a triangular set of differential polynomials

$H_A = i_1 \cdots i_r \, s_1 \cdots s_r$ the product of the initials and separants of A

Thanks to Ritt's partial reduction, the elements of A can be made **pairwise partially reduced** under the assumption $\dot{y} \neq 0$

$$A \left\{ \begin{array}{rcl} z & = & 2\,, \\ \dot{y}^2 & = & 4\,y\,, \end{array} \right. \qquad \dot{y} \neq 0\,. \tag{5}$$

Summary If the elements of A are pairwise partially reduced then every non differential zero of the following system can be prolongated into a differential zero

$$A \;=\; 0\,, \qquad H_A \;\neq\; 0\,.$$

Remark The system may have no solution

The Coherence Condition

In $\mathscr{F}\{y, z\}$ endowed with $\partial/\partial x$ and $\partial/\partial t$

The elements of A are pairwise partially reduced and $H_A = 1$ but A is not coherent

$$A \begin{cases} y_x &= z\,, \\ y_t &= 0 \end{cases}$$

Not every non differential zero of $A = 0$ can be prolongated to a differential zero since $z_t = 0$ (A is not coherent)

The Coherence Condition

In $\mathscr{F}\{y, z\}$ endowed with $\partial/\partial x$ and $\partial/\partial t$

System A can be made coherent

$$A \begin{cases} y_x = z\,, \\ y_t = 0\,, \\ z_t = 0 \end{cases}$$

Def z_t is a particular case of a Δ-polynomial

Test If all Δ-polynomials of A are reduced to zero by A then A is coherent

Summary If A is coherent and its elements are pairwise partially reduced then every non differential zero of following system can be prolongated into a differential zero

$$A = 0\,, \qquad H_A \neq 0\,.$$

Specification of the Differential Elimination Algorithm

Given an input system Σ and a ranking, it is possible to compute an equivalent set of finitely many regular differential chains

$$A_1, \ldots, A_\varrho$$

If this decomposition is empty ($\varrho = 0$) then $1 \in \{\Sigma\}$ (the perfect differential ideal) and Σ has no solution at all

If this decomposition is nonempty ($\varrho > 0$) then every non differential zero of

$$A_i = 0, \quad H_{A_i} \neq 0$$

can be prolongated into a differential zero of Σ

Hence there exist initial values for which Σ has formal power series solutions

Solving in Formal Power Series: the Difficult Case

$$p(x, y) \quad = \quad 0 \quad (\operatorname{ord} p = n)$$

Difficult cases arise when the separant $s(x, y) = \partial p / \partial y^{(n)}$ does not vanish identically but vanishes at the prescribed initial values

At some point, the existence problem of a formal power series solution amounts to find nonnegative integer solutions to polynomials in m variables (the number of derivations)

The PDE case mostly leads to undecidability results by Matiiassevich (1970) negative answer to Hilbert's 10th Problem

The ODE case ($m = 1$) seems to be algorithmic

J. Denef and L. Lipshitz. *Power Series Solutions of Algebraic Differential Equations*. 1984

A. Hurwitz. *Sur le développement des fonctions satisfaisant à une équation différentielle algébrique*. 1899

The Number of Initial Values Depends on the Initial Values

$$\begin{aligned} \bar{\bar{y}} &= y_0 + y_1\, x + y_2\, \frac{x^2}{2} + y_3\, \frac{x^3}{6} + \cdots \quad \text{solution of} \\ p(x, y) &= 0 \quad (\text{ord}\, p = n) \end{aligned}$$

The number of needed initial values depends on the initial values

The Number of Initial Values Depends on the Initial Values

$$\begin{aligned} \bar{\bar{y}} &= y_0 + y_1\,x + y_2\,\frac{x^2}{2} + y_3\,\frac{x^3}{6} + \cdots \quad \text{solution of} \\ p(x,y) &= 0 \quad (\operatorname{ord} p = n) \end{aligned}$$

Let us give ourselves infinitely many of them and encode them in a series

$$\bar{y} = y_0 + y_1\,x + y_2\,\frac{x^2}{2} + y_3\,\frac{x^3}{6} + \cdots \quad \text{(i.v. encoding series)}$$

The Number of Initial Values Depends on the Initial Values

$$\begin{aligned} \bar{\bar{y}} &= y_0 + y_1\,x + y_2\,\frac{x^2}{2} + y_3\,\frac{x^3}{6} + \cdots \quad \text{solution of} \\ p(x,y) &= 0 \quad (\operatorname{ord} p = n) \end{aligned}$$

Let us give ourselves infinitely many of them and encode them in a series

$$\bar{y} \;=\; y_0 + y_1\,x + y_2\,\frac{x^2}{2} + y_3\,\frac{x^3}{6} + \cdots \quad \text{(i.v. encoding series)}$$

From p and $\bar{y}$ compute
δ the number of needed initial values
β the size of the polynomial system to solve

$$p_0 = p_1 = \cdots = p_{\beta-1} = 0$$

Thm Every solution of the polynomial system (if any) can be prolongated into a formal power series solution $\bar{\bar{y}}$

Leading Coefficients are Now Given by a Polynomial $A(q)$

$$
\begin{aligned}
p(x,y) &= y\,\ddot{y} + \dot{y}^2 - 6\,y \\
s(x,y) &= y \\
\bar{y} &= y_2\,\frac{x^2}{2} + \cdots \quad (y_0 = y_1 = 0) \\
k &= 2 \quad (\text{the valuation of } s(x,\bar{y}),\ \text{assuming } y_2 \neq 0) \\
A(q) &= y_2\,q^2 + 15\,y_2\,q + 56\,y_2 - 12
\end{aligned}
$$

For $(y_0, y_1, y_2) = (0, 0, 2)$ we have $\beta = \delta = 8$

$$
\begin{array}{llcll}
p_3 & \frac{7}{3}\,y_3 & = & 0 & \frac{A(-3)}{3!} \\
p_4 & 1\,y_4 + \frac{5}{12}\,y_3^2 & = & 0 & \frac{A(-2)}{4!} \\
p_5 & \frac{3}{10}\,y_5 + \frac{7}{24}\,y_3\,y_4 & = & 0 & \frac{A(-1)}{5!} \\
p_6 & \frac{5}{72}\,y_6 + \frac{7}{144}\,y_4^2 + \frac{7}{90}\,y_3\,y_5 & = & 0 & \frac{A(0)}{6!} \\
p_7 & \frac{11}{840}\,y_7 + \frac{1}{40}\,y_4\,y_5 + \frac{1}{60}\,y_3\,y_6 & = & 0 & \frac{A(1)}{7!}
\end{array}
$$

We have found $y(x) = x^2$

An Undecidability Result

The problem: *does a given ordinary differential polynomial system have any nonzero formal power series solution centered at the origin?* is algorithmically undecidable [Singer, 1978]

Singer's argument relies on the following system

$$\begin{aligned} x\,\dot{y} &= \alpha\, y \\ \dot{\alpha} &= 0 \end{aligned}$$

which has a nonzero formal power series solution $y(x) = x^{\alpha}$ if and only if α is a nonnegative integer (Hilbert's 10th Problem)

Denef and Lipshitz formula permits to solve the existence and uniqueness problem of a formal power series solution under the assumption that we can determine k (over our example, we had to assume $y_2 \neq 0$)