TOPICS IN DIFFERENTIAL ALGEBRA

SEBASTIAN FALKENSTEINER

Contents

1. Differential elimination methods

In the course on differential algebra, we have seen the computation of regular differential chains for (not necessarily prime) differential ideals. Let us give a brief summary of the latter and introduce inequalities instead of using saturation.

Let us denote by $\mathcal{F}$ a differential field, possibly involving the independent variables $\mathbf{x} = (x_1, \ldots, x_m)$ such as it is the case for $\mathcal{F} = \mathbb{C}(\mathbf{x})$, and let $\mathcal{F}\{u_1, \ldots, u_n\} =: \mathcal{F}\{\mathbf{u}\}$ be the differential polynomial ring involving the differential operators $\frac{\partial}{\partial x_i}$. Let $\mathfrak{U} \subseteq \mathcal{F}\{\mathbf{u}\}$ be a differential ideal. Then there are prime differential ideals $\mathfrak{P}_i$ such that

$$\mathfrak{U} = \mathfrak{P}_1 \cap \ldots \cap \mathfrak{P}_\rho.$$

It is an unsolved problem to find a minimal prime decomposition, i.e., find $\mathfrak{P}_i$ such that none of them is included in another prime component.

For any given set $\Sigma \subset \mathcal{F}\{\mathbf{u}\}$, there exist finitely many regular differential chains $A_1, \ldots, A_\rho \subset \mathcal{F}\{\mathbf{u}\}$ such that

$$\{\Sigma\} = [A_1] : H_{A_1}^\infty \cap \cdots \cap [A_\rho] : H_{A_\rho}^\infty$$

where $H_{A_i}^\infty$ contains the initials and separants of A_i and $[A_i]$ denotes the differential ideal generated by A_i.

In the following, we will call a perfect differential ideal $\mathfrak{U} = [A] : H_A^\infty$, given by a single regular differential chain A, a *characterizable ideal.* Every perfect differential ideal is characterizable. Let us note that the saturation with H_A can be encoded by inequalities $H_A \neq 0$ added to the equalities in A. These inequalities are thus always reduced w.r.t. the equalities. The so-called *Thomas decomposition*, whose construction is very similar

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to the Rosenfeld-Gröbner algorithm besides replacing the triangularity of the system with passiveness [11], decomposes the solution space as

$$\mathrm{Sol}(\mathfrak{U}) = \dot{\bigcup}_{i\in\{1,\dots,\rho\}} \mathrm{Sol}(A_i),$$

where A_i are the regular differential chains in the Thomas decomposition consisting of equalities and inequalities. Here the solution set, denoted by Sol, could be formal power series (with a fixed expansion point), generic zeros, meromorphic functions etc. We will use the set of generic zeros, which are defined for the prime components of $\mathfrak{U}$, or formal power series centered at zero.

Example 1. *The system $\{p = u_{x_1,x_1}^2 + u_{x_1} + u_{x_2,x_2} + u_{x_2}\}$ simplifies to the Thomas decomposition (w.r.t. $u_{x_1} \gg u_{x_2}$)*

$$A_1 = \{u_{x_1,x_1}^2 + u_{x_1} + u_{x_2,x_2} + u_{x_2} = 0, u_{x_1} + u_{x_2,x_2} + u_{x_2} \neq 0\},$$
$$A_2 = \{u_{x_1,x_1} = 0, u_{x_1,x_2,x_2} + u_{x_1,x_2} = 0, u_{x_1} + u_{x_2,x_2} + u_{x_2} = 0\}.$$

Exercise 2. *Compute a regular (differential) chain of the parametric quadratic equation $a\,u^2 + b\,u + c = 0$ by considering the system*

$$\{p = a\,u^2 + b\,u + c, a', b', c'\} \subset \mathbb{C}\{a, b, c, u\}.$$

Give the answer in terms of saturation or inequalities.

Theorem 3 (Nullstellensatz; cf. Theorem 4 in DA)**.** *Let $\mathfrak{U} \subsetneq \mathcal{F}\{\mathbf{u}\}$ be a radical differential ideal. Then $\mathrm{Sol}(\mathfrak{U}) \neq \emptyset$. Moreover, if $p \in \mathcal{F}\{\mathbf{u}\}$ and $p(\mathbf{u}(\mathbf{x})) = 0$ for every $\mathbf{u}(\mathbf{x}) \in \mathrm{Sol}(\mathfrak{U})$, then $p \in \mathfrak{U}$.*

Example 4. *There is no Nullstellensatz for formal power series solutions with a fixed expansion point. In order to see this, consider $\{p = xu' - 1\}$ which has the generic zero $u(x) = c + \log(x)$ which cannot be expanded as formal power series around zero.*

2. Zero-testing

In this section, let us mainly focus on the ordinary case of one differential indeterminate and denote $\mathcal{F} = \mathbb{C}(x)$. Let $u(x) \in \mathbb{C}[[x]]$ be a formal power series. Then we call $u(x)$ *differentially algebraic* if there is $p \in \mathcal{F}\{u\} \setminus \mathcal{F}$ such that $p(u(x)) = 0$. The corresponding p is called the *annihilator* of $u(x)$ and we may assume w.l.o.g. that $S_p(u(x)) \neq 0$. If p can be chosen in $\mathcal{F}[u] \setminus \mathcal{F}$, i.e. does not involve any derivatives, then $u(x)$ is called *algebraic*. Let us note that it is not necessary that $u(x)$ is assumed to be a formal power series (with non-negative integer exponents). In fact, already algebraic formal power series might involve fractional exponents (sometimes also called *formal Puiseux series*). For differentially algebraic series, typically logarithms and exponentials in x appear.

Proposition 5. *The set of differentially algebraic formal power series form a ring closed under composition, division and differentiation, if they are defined.*

Proposition 6 (Proposition 2 in [12])**.** *A formal power series $u(x) \in \mathbb{C}[[x]]$ is differentially algebraic if and only if $\mathcal{F}\{u(x)\}$ has finite transcendence degree over $\mathcal{F}$.*

For uniquely representing a differentially algebraic formal power series, we have to specify which of the roots of a the annihilator is chosen.

Example 7. *The fomal series $u(x) = 1 + \log(x)$ is not algebraic but differentially algebraic with the annihilator $p = xu' - 1$. For $q = \exp(u) - 1$ we see that $q(u(x)) \neq 0$ but the conjugate root $v(x) = \log(x)$ fulfills $q(v(x)) = 0$.*

An often occuring case is when there are just exponentials or logarithms in $u(x)$ involved. Then the necessity of (algebraic) independence can often be shown by using the following result.

Theorem 8 (Theorem 1 in [1]). *Let $\mathbf{u}(\mathbf{x}) \in \mathbb{C}[[\mathbf{x}]]$ have $\mathbb{Q}$-linearly independent components. Then the transcendence degree of $\mathbb{Q}(\mathbf{u}(\mathbf{x}), \exp(u_1(\mathbf{x})), \dots, \exp(u_n(\mathbf{x})))$ (over $\mathbb{Q}$) is greater or equal to $n + \operatorname{rank}(\mathcal{J}(\mathbf{u}(\mathbf{x})))$, where $\mathcal{J}$ denotes the Jacobian-matrix w.r.t. $\mathbf{x}$, if and only if $u_1(\mathbf{x}) - u_1(0), \dots, u_n(\mathbf{x}) - u_n(0)$ are $\mathbb{Q}$-linearly independent.*

Example 9. *Let $\mathbf{u}(x_1, x_2) = (x_1 - x_2, x_1 x_2)$. Then $u_1(\mathbf{x}) - u_1(0), u_2(\mathbf{x}) - u_2(0)$ are $\mathbb{Q}$-linearly independent. The Jacobian-matrix*

$$\mathcal{J}(\mathbf{u}(\mathbf{x})) = \begin{pmatrix} 1 & -1 \\ x_2 & x_1 \end{pmatrix}$$

has rank 2 such that $\mathbb{Q}(x_1 - x_2, x_1 x_2, \exp(x_1 - x_2), \exp(x_1 x_2))$ has transcendence degree 4. Thus, there is no algebraic relation among $x_1 - x_2, x_1 x_2, \exp(x_1 - x_2)$ and $\exp(x_1 x_2)$.

Exercise 10. *Show that the logarithm $\log(1-x)$ is transcendental over the rational numbers attached by exponentials of polynomials, i.e., over $\mathbb{Q}(x, \exp(x), \exp(x^2), \dots, \exp(x^n))$ for every $n \in \mathbb{N}$.*
Is the logarithm $\log(1-x)$ differentially transcendental over $\mathbb{Q}(x)$?

Zero-testing is particularly important for us when considering a regular differential chain involving inequalities. In other words, there might be given an algebraic expression $q \in \mathcal{F}\{\mathbf{u}\}$ and a differentially algebraic formal power series $\mathbf{u}(\mathbf{x})$ and we want to test whether $q(\mathbf{u}(\mathbf{x})) = 0$. There are several different approaches for this and we focus here on two of them.

Structural relations. In some simple cases it is possible to relate all differentially algebraic formal power series under consideration. In this case, we just have to check whether the zero-test q is either among these relations or not. By using differential elimination for instance, we can find structural relations as follows. Let A be a differential regular chain and let $q \in \mathcal{F}\{\mathbf{u}\}$. If $\operatorname{prem}(q, A) = 0$, then q vanishes at every zero of A.

Example 11. *Let $A = \{p_1 = u_1^2 - x^3, p_2 = u_2^5 - x^3 u_1\}$ and $q = 8u_1'^3 - 27u_1$. Then $\operatorname{prem}(q, A) = 0$ and every zero of A, namely $\mathbf{u}(x) = (\pm x^{3/2}, \pm\zeta \cdot x^{9/10})$ where $\zeta^5 = 1$, is a solution of $q = 0$.*

Exercise 12. *Show that the converse is not true by using the regular differential chain A from Example 11 and finding $q \in \mathcal{F}\{\mathbf{u}\}$ with $\operatorname{prem}(q, A) \neq 0$ such that there is $\mathbf{u}(x) \in \mathbb{C}[[\mathbf{x}]]$ with $A(\mathbf{u}(x)) = q(\mathbf{u}(x)) = 0$.*

2.1. **Root separation bound.** We now want to choose a given root and find particular relations instead of structural relations which might be too general.

Let $u(x) \in \mathbb{C}[[x]]$ be differentially algebraic with annihilator $p \in \mathcal{F}\{u\}$. There exists a number $\rho \in \mathbb{N}$ such that for any $v(x) \in \mathbb{C}[[x]]$, $u(x) \neq v(x)$, with $p(v(x)) = 0$, it holds that $\operatorname{ord}_x(u(x) - v(x)) > \rho$. The smallest such number ρ will be called the *root separation* of p at $u(x)$. It corresponds to the number of initial conditions that should be known in order to determine $u(x)$ in a unique way as a root of p.

Proposition 13 (Proposition 4 in [12]). *There exists a bound $\rho_{p,u(x)}$ on the root separation of p at $u(x)$ in terms of the coefficients, exponents and order of p.*

Together with the result that for a given $u \in \mathbb{C}[[x]]$ and $q \in \mathbb{C}[x]\{u\}$ with $\operatorname{ord}_x(q(u(x))) > 2\rho_{q,u(x)}$ and $S_q(u(x)) \neq 0$, there exists a unique root $v(x) \in \mathbb{C}[[x]]$ of q with $\operatorname{ord}_x(u(x) - v(x)) > \rho_{q,u(x)}$, we obtain the following zero test.

Algorithm 1 ZeroTest(q)

Input: $u(x) \in \mathbb{C}[[x]]$ with annihilator $p \in \mathcal{F}\{u\}$ and $q \in \mathcal{F}\{u\}$
Output: true if $q(u(x)) = 0$ and false otherwise

If $q \in \mathbb{C}[x]$, return $q = 0$
If ZeroTest(I_q) then return ZeroTest(prem(q, I_q))
If ZeroTest(S_q) then return ZeroTest(prem(q, S_q))
If prem(q, p) $\neq 0$ then return ZeroTest(prem(q, p))
Return $\mathrm{ord}_x(q(u(x))) > 2\rho_{q,u(x)}$

Exercise 14. *Apply Algorithm 2.1 to* $u = \sqrt{1+x}$ *and*

(1) $q = 2uu' - 1$*;*
(2) $q = 2uu' - 1 + x^{10}/10^{10}$*;*
(3) $q = (2uu' - 1)u''^2 - xu + 1$.

The above theoretical results and Algorithm 2.1 can be generalized to several given $u_1, \ldots, u_n$ by using differential elimination methods and an iterative root seperation bound. Generalizations to $q_1, \ldots, q_m$ and the multivariate setting exist as well.

Zero-testing of power series involving parameters can be reduced to the case of several unknown functions. The problem of trying to directly apply Algorithm 2.1 is that the zero-bound ρ is in general not uniform in the parameters as it can be shown for the following equation.

Exercise 15. *Try to find a root-separation bound for* $p = xu' - cu$ *where* $c \in \mathbb{C}$ *is a parameter.*

2.2. **Heuristic zero-testing.** Zero-testing is a very asymmetric problem in the sense that in order to show that $q(u) \neq 0$ one just needs to find an order with a non-zero coefficient. Exact zero tests are rather slow, so it is often preferable to use heuristic zero tests instead. In particular, if a computation involves several zero-tests, one might first perform all zero-tests heuristically and only if it passes, the exact zero-tests are used.

The most obvious heuristic zero test for a univariate differentially algebraic series is to compute all coefficients up to a fixed order which is relatively efficient by using Newton's method or relaxed power series evaluation (see e.g. [2]).

In the multivariate case it is computationally preferable to reduce to zero-testing in the univariate case. This works in most cases by considering for given $u(\mathbf{x}) \in \mathcal{F}[[\mathbf{x}]]$ the series $u(\lambda \cdot \mathbf{x})$ for parameters $\lambda \in \mathcal{F}^m$.

Exercise 16. *Let* $u(x_1, x_2) = x_1 + x_2 - x_1 x_2$ *and consider*

(1) $q = u_{x_1} u_{x_2} + u - 1$*;*
(2) $q = u_{x_1}^2 + u - 1$.

Check with the above method whether $q(u(\mathbf{x})) = 0$.

3. Counting solutions of differential equations

In this section, we want to give a measure on the solution set of differential systems. First we start with purely algebraic systems.

A polynomial function $\omega : \mathbb{N} \to \mathbb{N}$ is called a *numerical polynomial*. On the set of numerical polynomials there is defined the total order $\omega_1 \leq \omega_2$ iff $\omega_1(\ell) \leq \omega_2(\ell)$ for all sufficiently large $\ell \in \mathbb{N}$.

Let $\mathfrak{U} \subset \mathcal{F}[\mathbf{u}]$ be an (algebraic) prime ideal. Then the *(algebraic) dimension* of $\mathfrak{U}$ is defined as the transcendence degree of $\mathcal{F}(\mathbf{u})/\mathfrak{U}$ over $\mathcal{F}$. This is computed by the elements "under the staircase" of a Gröbner basis of $\mathfrak{U}$ w.r.t. any chosen admissible ordering.

Let us denote by $\mathfrak{U}_\ell$ the elements of $\mathfrak{U}$ or degree less or equal to $\ell \in \mathbb{N}$. Then the *Hilbert function* is defined as

$$\Omega_{\mathfrak{U}} : \mathbb{N} \to \mathbb{N}, \ell \mapsto \dim(\mathcal{F}(\mathbf{u})_\ell / \mathfrak{U}_\ell).$$

It is well-known that there exists a unique numerical polynomial $\omega_{\mathfrak{U}}$, the so-called *Hilbert polynomial*, such that $\Omega_{\mathfrak{U}}(\ell) = \omega_{\mathfrak{U}}(\ell)$ for sufficiently large $\ell \in \mathbb{N}$. Moreover, the degree of $\omega_{\mathfrak{U}}$ coincides with $\dim(\mathfrak{U})$.

Example 17. *The algebraic ideal* $\mathfrak{U} = \langle u_1^3, u_1 u_2 \rangle \subset \mathbb{Q}[u_1, u_2]$*, which is already given by the Gröbner basis (w.r.t. any order)* $\{u_1^3, u_1 u_2\}$*, has the Hilbert function*

$$\Omega_{\mathfrak{U}}(0) = 1, \Omega_{\mathfrak{U}}(1) = 3, \Omega_{\mathfrak{U}}(\ell) = \ell + 3 \text{ for } \ell \geq 2.$$

Thus, the Hilbert polynomial is $\omega_{\mathfrak{U}}(\ell) = \ell + 3$ *and* $\dim(\mathfrak{U}) = 1$.

We are going to replicate the approach and results of the algebraic case to the differential case.

3.1. **Differential dimension.** In the proof of Proposition 16 (DA), we have seen how a *generic zero* of a prime differential ideal $\mathfrak{P}$ can be constructed. Recall that we make the ansatz

$$u_i(x_1, \ldots, x_m) = \sum_{\theta = \delta^{e_1} \cdots \delta^{e_m} \in \Theta} c_{i,\theta} (x_1 - \zeta_1)^{e_1} \cdots (x_m - \zeta_m)^{e_m}$$

for an unspecified center point ζ and unknown coefficients $c_{i,\theta}$ to which we assign iteratively values in $\mathcal{F}$. Note that this construction is independent of the representation of $\mathfrak{P}$ and can be performed also for non-prime differential ideals. The number of free coefficients $c_{i,\theta}$ can be seen as a measure for the dimension of the solution set. In the following, we will give formal definitions for this observation.

Let $\mathfrak{U} \subseteq \mathcal{F}\{\mathbf{u}\}$ be a differential ideal. We denote by $\mathfrak{U}_{\leq \ell}$ the elements of $\mathfrak{U}$ of order less or equal to ℓ. Then we define the map, by using the algebraic or Krull dimension and considering $u, u', \ldots$ as independent variables,

$$\Omega_{\mathfrak{U}} : \mathbb{N} \to \mathbb{N}, \ell \mapsto \dim(\mathcal{F}\{\mathbf{u}\}_{\leq \ell} / \mathfrak{U}_{\leq \ell}).$$

We call $\Omega_{\mathfrak{U}}$ the *differential dimension function* of $\mathfrak{U}$.

Proposition 18. *Let* $\mathfrak{U}$ *be a characterizable differential ideal with corresponding regular differential chain* A *with set of leading derivatives* $\operatorname{lead}(A)$*. Then, for every* $\ell \in \mathbb{N} \cup \{\infty\}$,

$$\Theta\{\mathbf{u}\}_{\leq \ell} \setminus \Theta \operatorname{lead}(A)_{\leq \ell} = \dim(\mathcal{F}\{\mathbf{u}\}_{\leq \ell} / \mathfrak{U}_{\leq \ell}).$$

Theorem 19 (Theorem 1.1 in [9])**.** *Let* $\mathfrak{U} \subseteq \mathcal{F}\{u_1, \ldots, u_n\}$ *be a characterizable differential ideal.*

(1) There exists a numerical polynomial $\omega_{\mathfrak{U}} \in \mathbb{Q}[\ell]$*, called differential dimension polynomial, with* $\Omega_{\mathfrak{U}}(\ell) = \omega_{\mathfrak{U}}(\ell)$ *for sufficiently large* $\ell \in \mathbb{N}$.

(2) $0 \leq \omega_{\mathfrak{U}}(\ell) \leq n \cdot \binom{\ell+m}{m}$*. In particular,* $\deg(\omega_{\mathfrak{U}}) \leq n$ *and we can write* $\omega_{\mathfrak{U}}(\ell) = \sum_{i=1}^{m} a_i \cdot \binom{\ell+i}{i}$ *for some* $a_i \in \mathbb{Z}$.

(3) The coefficient a_m *is the differential dimension and equal to the cardinality of the differential transcendence basis of* $\mathcal{F}\{u_1, \ldots, u_n\}/\mathfrak{U}$.

(4) Let $\mathfrak{U} \subseteq \mathfrak{V} \subset \mathcal{F}\{\mathbf{u}\}$ *be another characterizable differential ideal (w.r.t. the same ranking). Then* $\omega_{\mathfrak{U}} \leq \omega_{\mathfrak{V}}$*. Moreover,* $\mathfrak{U} = \mathfrak{V}$ *if and only if*
- $\omega_{\mathfrak{U}} = \omega_{\mathfrak{V}}$;
- *the set of leaders of the corresponding regular chains coincide; and*
- *the equations have the same degrees in their leaders.*

For differential ideals generated by algebraic polynomials, the differential dimension is equal to the algebraic dimension. In the case of ordinary differential equations ($m = 1$), $\omega_{\mathfrak{U}}(\ell)$ is affine-linear, i.e. of the form $a_1\,\ell + a_0$, and the differential dimension a_1 is non-zero if and only if no solution component u_i can be chosen freely. This is the case if and only if for every u_i there is a derivative $\theta_i u_i$ appearing as leader in the corresponding regular differential chain.

Let us now actually compute the differential dimension polynomial.

Proposition 20. *The differential dimension polynomial and the differential dimension of a characterizable differential ideal are independent of the the chosen orderly ranking and regular differential chain.*

For every $v \in \operatorname{lead}(A)$ we can associate a cone C_v in $\Theta\{u_1, \dots, u_n\} \setminus \Theta\operatorname{lead}(A)$ such that

$$\dot{\bigcup}_{v\in\operatorname{lead}(A)} C_v = \Theta\{u_1, \dots, u_n\} \setminus \Theta\operatorname{lead}(A).$$

The set $C = \{C_v \mid v \in \operatorname{lead}(A)\}$ is called a *Janet decomposition* of A.

Theorem 21. *Let $\mathfrak{U} \subseteq \mathcal{F}\{u_1, \dots, u_n\}$ be a characterizable differential ideal with corresponding regular differential chain A (w.r.t. an orderly ranking) and let C be a Janet decomposition of A. Then*

$$\omega_{\mathfrak{U}}(\ell) = n \cdot \binom{\ell + m}{m} - \sum_{v\in\operatorname{lead}(A)} \binom{\dim(C_v) + \ell - \operatorname{ord}(v)}{\dim(C_v)}.$$

Example 22. *In item (4) in Theorem 19, the second point in the reverse direction can be neglected (it follows from the first point), but the third one can not. In order to see this, let $A_1 = \{u_1\}$ and $A_2 = \{u_1^2 - u_1\}$. Then the differential ideals $\mathfrak{U} = \{A_2\} \subsetneq \mathfrak{V} = \{A_1\}$ are characterizable with regular differential chains A_2 and A_1, respectively. The corresponding differential dimension polynomials, however, are both constantly zero.*

Example 23. *The Burger's equation $p = u_{x_1,x_1} - u_{x_2} - 2uu_{x_1}$ defines a regular differential chain (w.r.t. any ranking). By using Theorem 21, we obtain $\omega_{\{p\}}(\ell) = 2\ell + 1$.*

It is important to use an orderly ranking for the computation of A. Otherwise the differential polynomial might not be the same for every choice of the regular differential chain. For instance, in case of the Burger's equation, if $u_{x_2} > u_{x_1,x_1}$, then we would obtain $\omega_{\{p\}} = \ell + 1$.

Exercise 24. *Try a software-system of your choice to compute solutions of the Burger's equation. Does the output cover all solutions?*

3.2. **Differential counting polynomial.** We now want to precisely count the formal power series solutions of differential equations by using the algebraic counting polynomial for the set of Taylor polynomials of degree ℓ for each $\ell \in \mathbb{N}$. The sequence of these algebraic counting polynomials is called the differential counting function, and we seek to give them by a closed formula in ℓ, the *differential counting polynomial*.

As we have seen in the course (DA), computing formal power series solutions of (systems of) algebraic differential equations can be transformed to the problem of solving infinite polynomial systems, possibly involving polynomial inequalities. Thus, we first want to understand the (finite) algebraic case.

The algebraic counting polynomial can be directly read-off a Thomas decomposition by considering the degrees of the leaders.

Example 25. *The algebraic ideal given by*

$$p = u^3 + (3v+1)u^2 + (3u^2+2u)v + u^3$$

has the Thomas decomposition (w.r.t. $u > v$)

$$\begin{aligned} A_1 &= \{u^3 + (3u+1)v^2 + (3v^2+2v)u + v^3 = 0, 27v^3 - 4v \neq 0\}, \\ A_2 &= \{6u^2 + (-27v^2+12v+6)u - 3v^2 + 2v = 0, 27v^3 - 4v = 0\}. \end{aligned}$$

System A_1 implies that for every v-value in $\mathcal{F}$, except the 3 solutions of $27v^3 - 4v = 0$, there are 3 solutions (in $\overline{\mathcal{F}}$) of $p = 0$. For each exceptional solution, by considering A_2, we find 2 solutions. Hence, the algebraic counting polynomial is $(\infty - 3) \cdot 3 + 3 \cdot 2 = 3\infty - 3$.

No inequalities. The number of zeros of a given characterizable differential ideal $\mathfrak{U}$ with a regular differential chain A involving no inequalities is measured, similarly to the algebraic case, as

$$C_{\mathfrak{U}} : \mathbb{N} \to \mathbb{Z}[\infty], \ell \mapsto \prod_i \deg_{\mathrm{lead}(A)}(p_i) \cdot \infty^{\omega_{\mathfrak{U}}(\ell)}$$

where ∞ is a formal indeterminate of the polynomial ring $\mathbb{Z}[\infty]$ which might be interpreted as the cardinality of $\overline{\mathcal{F}}$.

Example. *The Burger's equation $p = u_{x_1,x_1} - u_{x_2} - 2uu_{x_1}$ has the differential counting polynomial $C_{\{p\}}(\ell) = \infty^{2\ell+1}$.*

Example 26. *The incompressible Navier-Stokes equations are given by the differential system Σ consisting of*

$$\begin{aligned} p_1 &= u_t + uu_{x_1} + vu_{x_2} + wu_{x_3} + p_{x_1} - u_{x_1,x_1} - u_{x_2,x_2} - u_{x_3,x_3} = 0, \\ p_2 &= v_t + uv_{x_1} + vv_{x_2} + wv_{x_3} + p_{x_2} - v_{x_1,x_1} - v_{x_2,x_2} - v_{x_3,x_3} = 0, \\ p_3 &= w_t + uw_{x_1} + vw_{x_2} + ww_{x_3} + p_{x_3} - w_{x_1,x_1} - w_{x_2,x_2} - w_{x_3,x_3} = 0, \\ p_4 &= u_{x_1} + v_{x_2} + w_{x_3} = 0. \end{aligned}$$

A regular differential chain of Σ (w.r.t. an orderly ranking where $u > v > w > p$ and $u_{x_1} > u_{x_2} > u_{x_3} > u_t$) is found by adding the so-called Poisson pressure equation

$$p_5 = 2u_{x_2}v_{x_1} + 2u_{x_3}w_{x_1} + 2v_{x_3}w_{x_2} + u_{x_1}^2 + v_{x_2}^2 + w_{x_3}^2 + p_{x_1,x_1} + p_{x_2,x_2} + p_{x_3,x_3} = 0$$

and does not contain any inequalities. The differential dimension polynomial is $\omega_{\{\Sigma\}}(\ell) = 7\ell^3/6 + 11\ell^2/2 + 25\ell/3 + 4$. Thus, the differential counting polynomial is

$$C_{\{\Sigma\}}(\ell) = \infty^{7\ell^3/6+11\ell^2/2+25\ell/3+4}.$$

General systems. For a given radical differential ideal $\mathfrak{U}$ and its regular differential chains $A_1, \ldots, A_\rho$, the differential counting polynomial, if it exists, is the sum of their differential counting polynomials subtracted by polynomials smaller or equal to the differential counting polynomials corresponding to the inequalities. The addition is valid because of the disjointness of solutions. The subtraction depends on how many solutions of the equalities are removed.

To compute the differential counting polynomial, we compute the formal power series solutions (around an arbitrary center point ζ). In principle, the number of possibilities for the next coefficient of a formal power series solution could be anything from zero to infinity and a detailed analysis is necessary.

Example 27. *Compute the differential counting polynomial of the differential ideals given by*

(1) $p = u'^2 - 1$.
Although p *can be factored as a polynomial,* $\{p = 0\}$ *is already a regular differential chain without an inequality. Thus, by Theorem 21,* $\omega_{\{p\}}(\ell) = 1$ *and* $C_{\{p\}}(\ell) = 2\infty$.

(2) $p = u^2 u' - u$.
We see that $\{uu' - 1 = 0, u \neq 0\}, \{u = 0\}$ *is a system of regular differential chains. For the second system we count 1 solution. For the first system, we obtain the algebraic system*

$$\mathcal{S} := \{u_0 u_1 - 1 = 0, u_0 u_2 + u_1^2 = 0, u_0 u_3 + 2u_1 u_2 = 0, \ldots; u_0 \neq 0\}$$

and no system with $u_0 = 0$ *because this directly leads to a contradiction.* $\mathcal{S}$ *has for every* $u_0 \in \mathcal{F}$ *a unique solution. Thus, we count* $\infty - 1$ *and obtain for* $\{p = 0\}$ *the differential counting polynomial* $C_{\{p\}}(\ell) = 1 + \infty - 1 = \infty$. *Indeed, we have* $u(x) = 0$ *and* $v(x) = \pm\sqrt{2(x + c)}$ *as solutions and* $v(x)$ *can be expanded (counterinuitively) as a single formal power series.*

Note that the system (2) does not fulfill the assumption that it is characterizable nor that no inequality is involved, but when naively applying the formula from Theorem 21, the result is still correct. This will not be the case in the following example.

Exercise 28. *Compute the differential counting polynomial of* $p = u'^2 - u$.

Exercise 29. *Try to find the differential counting polynomial of a differential system given by a single equation of the form* $p = a(u)u' + b(u)$ *with* $a, b \in \mathcal{F}[u]$.

Differential counting polynomial with fixed center. We now want to count the number of formal power series solutions with an a-priori fixed center point ζ. Similarly as before, the differential counting function of a differential ideal $\mathfrak{U}$ is defined as the number of possible Taylor-coefficients of truncated formal power series solutions (centered around ζ) and is eventually represented by a polynomial function $C_{\zeta,\mathfrak{U}}(\ell)$ for large enough ℓ. Within this section we choose w.l.o.g. $\zeta = 0$.

Let us note that if Σ has constant coefficients, then the computation of the formal power series solutions centered at ζ and formal power series solutions with an unspecified center point coindes and thus, its differential counting polynomial coincide.

Example 30. *Let us consider* $p = vu' - u$ *and a ranking* $u > v$. *Let us write* $u(x) = \sum_{i \geq 0} c_i x^i$, $v(x) = \sum_{i \geq 0} d_i x^i$. *For* $d_0 \neq 0$, *every* d_i *is uniquely determined by the* $c_i \in \mathbb{C}$. *The* c_i *are completely unconstrained leading to the counting of* $\ell \mapsto (\infty - 1)\infty^{\ell+1}$. *For* $d_0 = 0$, *by simply plugging into* p *evaluated at* $x = 0$, *we see that* $c_0 = 0$. *By direct computation we see that (cf. Hurwitz lemma)*

$$p^{(k)} = \sum_{i=0}^{k} \binom{k}{i} v^{(i)} u^{(k-i+1)} - u^{(k)} = vu^{(k+1)} + (kv' - 1)u^{(k)} + \cdots .$$

Evaluated at $x = 0$, *we obtain either*

$$\mathcal{S}(k) := \{p_k = 0, c_0 = \cdots = c_{k-1} = 0, d_0 = 0, \prod_{i=1}^{k-1}(id_i - 1) \neq 0\}, \qquad \textit{or}$$
$$\mathcal{T}(k) := \{p_{k+1} = 0, c_0 = \cdots = c_{k-1} = 0, d_0 = kd_1 - 1 = 0\}.$$

The case where in every step k *the system* $\mathcal{S}(k)$ *is chosen ("*$\mathcal{T}(\infty) = \cup_{k \geq 1} \mathcal{S}(k)$*") is where all coefficients of* $u(x)$ *are zero. The distinct cases* $\mathcal{T}(k)$ *lead to a disjoint solution set. Thus, we add*

$$\ell \mapsto (\infty - \ell)\infty^{\ell-1} + \sum_{j \geq 1}^{\ell} \infty^{\ell} = (\ell + 1)\infty^{\ell} - \ell\infty^{\ell-1}$$

where the first summand corresponds to $\cup_{j=1}^{\ell}\mathcal{S}(j)$ *and the second summand to* $\mathcal{T}(1),\ldots,\mathcal{T}(\ell)$. *Then the differential counting polynomial is*

$$C_{0,\{p\}}(\ell) = \infty^{\ell+2} - \infty^{\ell+1} + (\ell+1)\infty^{\ell} - \ell\infty^{\ell-1} + 1$$

for $\ell \geq 1$ *and* $C_{0,\{p\}}(0) = \infty^2 - \infty + 1$.

The following example is particularly interesting because it gives a (countable) infinite number of exceptional values, which might be encoded by subtracting $\aleph_0$ in the differential counting polynomial.

Exercise 31. *Compute the differential counting polynomial* $C_{0,\{p\}}$ *of* $p = xuu' - xu + 1$.

From [4] we know that there is no general decision algorithm for deciding whether a system of algebraic differential equations has formal power series solutions. In particular, this shows that in general it is impossible to count the number of formal power series solutions and find the differential counting polynomial algorithmically.

4. Representations of differential equations

Algebraic varieties are represented as the zero set of polynomials. In some cases, we can find an explicit representation of them by rational / radical / etc. parametrizations. For differential systems, an analogue are so-called realizations. Such as parametrizations allow the computation of zeros by simply plugging in numbers, realizations help for finding zeros of the given differential system.

4.1. **Realizations.** In this section, we consider a class of systems of differential equations ubiquitous in applications. More precisely, we will consider systems of the form

$$\begin{cases} \mathbf{t}' = \mathbf{f}(\mathbf{c},\mathbf{t},\mathbf{y}) \\ \mathbf{u} = \mathbf{g}(\mathbf{c},\mathbf{t},\mathbf{y}) \end{cases}, \tag{1}$$

where

- $\mathbf{c}$ are unknown parameters;
- $\mathbf{t},\mathbf{u},\mathbf{y}$ are function variables, denpending on the independent variable x, referred to as the state, output, and input variables, respectively;
- $\mathbf{f},\mathbf{g}$ are vectors of polynomials in $\mathbb{C}(\mathbf{c})[\mathbf{t},\mathbf{y}]$.

In the following we will often neglect the dependency onto the parameters $\mathbf{c}$. The first equation $\mathbf{t}' = \mathbf{f}(\mathbf{c},\mathbf{t},\mathbf{y})$ is called a *dynamical system*. System (1) can be interpreted as follows. The input variables $\mathbf{y}$ are the functions determined by the experimenter (e.g., an external force). Together with the parameter values and the initial conditions for the state variables $\mathbf{t}$, they completely define the dynamics of the $\mathbf{t}$-variables. The output variables $\mathbf{u}$ are the quantities observed in the experiment. The typical questions asked about such systems include:

- When is an explicit representation of a given differential system possible?
- Is it possible to determine the values of the parameters (identifiability) or reconstruct the values of the state variables (observability)?
- Is it always possible to achieve the desired behaviour of the system by chosing appropriate input functions (controllability)?

To put (1) into the context of differential algebra, consider the purely transcendental extension $\mathbb{C}(\mathbf{c})$ as base field and the differential polynomials

$$Q(\mathbf{t},\mathbf{y})\cdot\mathbf{t}' - \mathbf{F}(\mathbf{t},\mathbf{y}), Q(\mathbf{t},\mathbf{y})\cdot\mathbf{u} - \mathbf{G}(\mathbf{t},\mathbf{y}) \in \mathbb{C}(\mathbf{c})\{\mathbf{t},\mathbf{y},\mathbf{u}\}$$

where $f_i = F_i/Q, g_i = G_i/Q$.

Proposition 32 (Lemma 3.2 in [7])**.** $\mathfrak{P} := [Q(\mathbf{t},\mathbf{y})\cdot\mathbf{t}' - \mathbf{F}(\mathbf{t},\mathbf{y}), Q(\mathbf{t},\mathbf{y})\cdot\mathbf{u} - \mathbf{G}(\mathbf{t},\mathbf{y})] : Q^\infty$ *forms a prime differential ideal.*

Input-output equations. Based on Proposition 32, we can compute a regular differential chain A of $\mathfrak{P}$ with a ranking $\mathbf{t} > (\mathbf{u},\mathbf{y})$. Then $A \cap \mathbb{C}(\mathbf{c})\{\mathbf{y},\mathbf{u}\}$ are called the *input-output equations* of (1) and generate $\mathfrak{P} \cap \mathbb{C}(\mathbf{c})\{\mathbf{y},\mathbf{u}\}$.

Example 33 (Predator-prey model)**.** *The following model describes the coexistence of two species, prey (t_1) and predators (t_2), so that the population of prey can be observed and controlled:*

$$\begin{cases} t_1' = c_1t_1 - c_2t_1t_2 + y \\ t_2' = -c_3t_2 + c_4t_1t_2 \\ u = t_1 \end{cases}.$$

By choosing $\mathbf{t} > \mathbf{y} > \mathbf{u}$ *we obtain the regular differential chain*

$$A = \{uu'' + c_1c_4u^3 - c_1c_3u^2 - c_4u^2u' + c_3uu' - u'^2, -t_1 + u, -c_2ut_2 - c_1u - u' + y\}.$$

The first differential polynomial of A is the input-output equation.

Realization problem. Let us now try to find for a given system $A \subset \mathbb{C}(\mathbf{c})\{\mathbf{y},\mathbf{u}\}$ of differential equations a corresponding system Σ as in (1). If such Σ exists, then we call it a *realization* of A.

For a given differential system Σ as in (1) with one output-equation u, there exists a single input-output equation which is unique up to multiplication with constants. Conversely, for a given irreducible differential polynomial $p \in \mathbb{C}(\mathbf{c})\{\mathbf{y},\mathbf{u}\}$ (or a system of input-output equations), there might be various realizations or none at all.

Let $\Sigma \subset \mathbb{K}[u_0,\dots,u_n]$. Then the variety given by Σ is given as

$$\mathbb{V}(\Sigma) = \{a \in \overline{\mathbb{K}}^{n+1} \mid p(a) = 0 \text{ for every } p \in \Sigma\}.$$

Let $p \in \mathbb{C}(\mathbf{c})\{y,u\}$ be a given irreducible differential polynomial of order n w.r.t. u. If p has a realization $\mathbf{t}' = \mathbf{f}(\mathbf{t},y), u = g(\mathbf{t},y)$, then

$$\mathcal{P} = (g, \mathcal{L}_\mathbf{f}(g), \dots, \mathcal{L}_\mathbf{f}^n(g)), \tag{2}$$

where $\mathcal{L}_\mathbf{f}(g) = \sum_{i=1}^n f_i\,\partial_{t_i}g + D_y(g)$ is the Lie-derivative of g w.r.t. $\mathbf{f}$, $\mathcal{L}_\mathbf{f}^i$ is the iterative application of $\mathcal{L}_\mathbf{f}$ i-many times, and D_y is defined as the differential operator $D_y(g) = \sum_{j\geq 0} y^{(j+1)}\cdot\partial_{y^{(j)}}g$, defines a parametrization of $\mathbb{V}(p)$. Thus, we have found a necessary condition for the existence of realizations. Note that the construction of the parametrization (2) from the realization does not require field extensions, keeps polynomiality of $\mathbf{f}, g$ etc. For a system of input-output equations in several output-variables, a similar construction can be made.

Proposition 34. *Let $p \in \mathbb{K}\{y,u\}$ be an irreducible polynomial of order n w.r.t. u. Then, p is realizable if and only if there is a rational parametrization $\mathcal{P} \in \overline{\mathbb{K}}(y,\dots,y^{(n)})(\mathbf{t})^{n+1}$ of $\mathbb{V}(p)$ such that $P_0 \in \overline{\mathbb{K}}(y,\mathbf{t})$ and*

$$\mathbf{z} \;=\; \mathcal{J}(P_0,\dots,P_{n-1})^{-1}\cdot(P_1 - D_y(P_0),\dots,P_n - D_y(P_{n-1}))^T \tag{3}$$

is in $\overline{\mathbb{K}}(y,\mathbf{t})^n$ where $\mathcal{J}$ denotes the Jacobian (w.r.t. $\mathbf{t}$). In the affirmative case, the realization is $\mathbf{t}' = \mathbf{z}, u = P_0$.

Let us note that if in Proposition 34 the polynomial p is independent of y, then (3) is also independent of y and its derivatives and thus, a rational parametrization $\mathcal{P}$ of $\mathbb{V}(p)$ always provides a realization.

Exercise 35. *Check whether the following input-output equations are realizable.*

(1) $p = u^2 + u'^2 - 1$;
(2) $p = (u' - yu)^3 + yu^2$;
(3) $p = 2u^2u'^2 + u^2 + 2u'^2$.

Exercise 36. *Let $p \in \mathbb{K}\{u, y\}$ be an irreducible polynomial of order n w.r.t. u. Show that if the order of p w.r.t. y is bigger than n, then p is not realizable.*

Identifiability. Let $(\tilde{\mathbf{y}}, \tilde{\mathbf{u}})$ be the general solution of the input-output equations of a given system as in (1). Then a function $h \in \mathbb{C}(\mathbf{c})$ is called *identifiable* iff h is in the smallest differential subfield of $\mathbb{C}(\mathbf{c})$ generated by $(\tilde{\mathbf{y}}, \tilde{\mathbf{u}})$. Of particular interest is the case where a parameter c_i itself is identifiable.

The smallest field $\mathbb{K} \subseteq \mathbb{C}(\mathbf{c})$ such that the input-output equations of A lie in $\mathbb{K}\{\mathbf{y}, \mathbf{u}\}$ is called its *field of definition*. The field generated by the identifiable functions is equal to the field of definition [10, Theorem 11] and thus can be read-off the coefficients of the input-output equations.

Example. *In our predator-prey model we obtain the identifiable functions*

$$\mathbb{C}(c_1c_4, c_4, c_1c_3, c_3) = \mathbb{C}(c_1, c_3, c_4)$$

meaning that the values of c_1, c_3, c_4 can be inferred from a series of experiments while the value of c_2 cannot.

Exercise 37. *Determine the identifiable functions of the given dynamical system*

$$\begin{cases} t_1' = c_1c_2t_1 - t_1t_2 + y \\ t_2' = -t_2 + c_3t_1t_2 \\ u = t_1 \end{cases}.$$

4.2. **Finding solutions.** A naive approach for finding rational solutions of a given (system of) algebraic differential equations is to make an ansatz with unknown coefficients and then derive necessary conditions for them. For algebraic solutions, we can apply a similar approach by first computing any local solution and then check whether it is algebraic. In general, however, we do not have any degree bound for rational solutions of algebraic differential equations. Furthermore, if the degree of the solutions is relatively high, the computations for the coefficients is very costly and will not terminate in reasonable time.

Rational solutions. Let us come back to differential ideals given by a set of differential polynomials $A = \{p_1, \ldots, p_m\} \subset \mathbb{K}\{u_1, \ldots, u_n\}$ independent of input-variables. We have seen that if there is a rational parametrization $\mathcal{P}$ of the associated algebraic variety $\mathbb{V}(A)$, then we can transform A into a dynamical system Σ. Moreover, solutions can generically be transformed into each other.

Proposition 38. *Let $\mathbf{u}(x) \in \overline{\mathbb{K}}(x)^n$ be a zero of a given set of differential polynomials A such that $\mathbb{V}(A)$ admits a rational parametrization $\mathcal{P}$. Then one of the following holds.*

(1) $\mathbf{u}(x) = \mathcal{P}(\mathbf{t}(x))$ *where $\mathbf{t}(x)$ is a solution of the corresponding dynamical system.*
(2) $\mathbf{u}(x) \notin Im(\mathcal{P})$.

Moreover, every solution $\mathbf{t}(x)$ of the corresponding dynamical system leads to the zero $\mathcal{P}(\mathbf{t}(x))$ of A.

The second item is the non-generic one (for rational / algebraic solutions $\mathbf{u}(x)$), since every rational parametrization has a dense image. This implies that for a rational generic solution of a given differential system and the generic rational solution of the corresponding dynamical system item (1) is fulfilled and they can always be transformed into each other.

Let us analyse the case of a single differential polynomial of order one with constant coefficients.

Theorem 39 (Theorem 5 in [6]). *Let $p \in \mathbb{K}[u, u']$ be an irreducible polynomial such that $\mathbb{V}(p)$ admits the birational parametrization $\mathcal{P} = (P_0, P_1) \in \overline{\mathbb{K}}(t)^2$. Then the corresponding dynamical system $t' = P_1(t)/P_0'(t)$ has a solution $t \in \overline{\mathbb{K}}(x)$ if and only if P_1/P_0' is a constant or of the form $b \cdot (t-a)^2$ for some $a, b \in \overline{\mathbb{K}}$.*

Example 40. *Let*

$$p = u'^3 + 4u'^2 + (27u^2 + 4)u' + 27u^4 + 4u^2.$$

The birational parametrization $\mathcal{P} = (216t^3 + 6t, -3888t^4 - 36t^2)$ fulfills Theorem 39, since $P_1/P_0' = -6t^2$, and leads to the rational generic solution

$$u(x) = \frac{(x+c)^2 + 1}{(x+c)^3}.$$

Exercise 41. *Let $p = u'^2 + 3u' - 2u - 3x$. Find a rational parametrization $\mathcal{P}$ of $\mathbb{V}(p)$ and its corresponding dynamical system. Can you find a rational generic solution of it? Is the solution $u_s(x) = -3x/2 - 9/8$ covered by the rational generic solution. Is $u_s(x)$ in the image of $\mathcal{P}$ (cf. Proposition 38)?*

Algebraic solutions. Given a differential system A in one differential unknown function u, if an irreducible polynomial $q(x, u)$ fulfills $\text{prem}(q, A) = 0$, then every zero of q is a zero of A. Here, an ansatz of unknown coefficients for q does in general not lead to a polynomial system in these coefficients. So we have to come up with another idea. In the case of first-order differential equations with constant coefficients, the following holds.

Theorem 42. *Let $p \in \mathbb{K}[u, u']$ and let $u(x) \in \overline{\mathbb{K}(x)}$ be an algebraic zero of p with minimal polynomial $q(x, u) \in \overline{\mathbb{K}}[x, u]$. Then all formal power series solutions of $p = 0$ are algebraic and given by $q(x + c, u)$, where $c \in \overline{\mathbb{K}}$.*

Theorem 43. *Let $p \in \mathbb{K}[u, u']$ and let $q(x, u) \in \overline{\mathbb{K}}[x, u]$ be the minimal polynomial of an algebraic zero of p. Then*

$$\deg_x(q) = \deg_{u'}(p), \, \deg_u(q) \le \deg_u(p) + \deg_{u'}(p).$$

The previous theorems imply an algorithm for computing algebraic solutions of first-order differential equations with constant coefficients.

Example 44. *Let $p = u^4 + 3u'$. We see that $p = 0$ cannot have a rational solution. A local solution is easily found by*

$$u(x) = 1 - x/3 + 2x^2/9 - 14x^3/81 + \mathcal{O}(x^4).$$

Let $q(x, u) = \sum_{0 \le i \le 4, 0 \le j \le 1} c_{i,j} x^i u^j$. Then $q(x, u(x)) = 0$ leads to the possible choice $q = xu^3 - 1$ and the zeros, namely $u(x) = \frac{\zeta}{\sqrt[3]{x+c}}$ for $\zeta^3 = 1$, are determined by $q(x + c, u)$.

The previous approach can be tried for more general differential equations, but in general it is unknown which local solution should be used for finding the algebraic solution and no general degree bounds for its minimal polynomial are known.

For global solutions such as rational or algebraic solutions, in contrast to local solutions, one can try to incorporate boundary conditions etc. Besides avoiding the question of zero-testing, this is one of the main advantages of global solutions compared to local solutions, if they exist and can be found. In the next section, we deal with more general local solutions than formal power series (with non-negative integer exponents).

Local solutions. In the case where $\mathbb{V}(p)$ does not admit a rational parametrization, we can still compute local parametrizations of the curve and derive local solutions if they exist. For this purpose, however, we do not have a corresponding dynamical system with rational right hand sides. Instead, we have to work with a formal power series right hand side and call this system the *associated differential equation.*

A *local parametrization* of an algebraic curve $\mathbb{V}(p)$, implicitly defined by $p \in \mathbb{K}[u_0, u_1]$, is a zero $(P_0, P_1) \in \overline{\mathbb{K}}[[t]]^2 \setminus \overline{\mathbb{K}}^2$ of p.

A power series with fractional exponents (with bounded denominator and a least element), i.e. series of the form $\sum_{i \geq i_0} c_i x^{i/n}$ with $i_0 \in \mathbb{Z}, n \geq 1$, is called a *formal Puiseux series.* Formal Puiseux series form a field which is algebraically closed for algebraically closed base fields $\overline{\mathbb{K}}$.

Theorem 45 (Theorem 10 in [3])**.** *Let $p \in \mathbb{K}[u, u']$ be an irreducible polynomial and let $\mathcal{P} = (P_0, P_1) \in \overline{\mathbb{K}}(t)^2$ be a local parametrization of $\mathbb{V}(p)$. Let*

$$o := \operatorname{ord}_t(P_0 - P_0(0)) - \operatorname{ord}_t(P_1) > 0.$$

Then the associated differential equation

$$P_0'(t)\, t' = o\, x^{o-1}\, P_1(t)$$

has a solution $t \in \overline{\mathbb{K}}[[x]]$, $t(0) = 0$, if and only if $y(x) = P_0(t(x^{1/o}))$ is a formal Puiseux series solution of $p = 0$.

Example 46. *Let us consider*

$$p = ((u' - 1)^2 + u^2)^3 - 4(u' - 1)^2 u^2 = 0.$$

The generic solutions is given by $u(x) = c_0 + c_1 x + \mathcal{O}(x^2) \in \mathbb{Q}(c_0, c_1)[[x]]$ with $p(c_0, c_1) = 0, S_p(c_0, c_1) \neq 0$. One of the critical curve points of $\mathbb{V}(p)$, not covered by the generic solution, is $(0, 1)$. One of the four local parametrizations centered at $(0, 1)$ is given by

$$(P_0, P_1) = (t^2, 1 + \sqrt{2}t - \frac{3t^3}{4\sqrt{2}} - \frac{15t^5}{64\sqrt{2}} + \mathcal{O}(t^6)).$$

The associated differential equation

$$tt' = x(1 + \sqrt{2}t - \frac{3t}{4\sqrt{2}} - \frac{15t^5}{64\sqrt{2}})$$

has the formal power series solutions $t_1(x) = x + \frac{\sqrt{2}x^2}{3} + \frac{x^3}{18} - \frac{89x^4}{540\sqrt{2}} + \mathcal{O}(x^5)$ and $t_2(x) = -x + \frac{\sqrt{2}x^2}{3} - \frac{x^3}{18} - \frac{89x^4}{540\sqrt{2}} + \mathcal{O}(x^5)$. Then $P_0(t_1(x^{1/2})) = x = \frac{2\sqrt{2}x^{3/2}}{3} + \mathcal{O}(x^{5/2}), P_0(t_2(x^{1/2}))$ are formal Puiseux series solutions of $p = 0$.

Exercise 47. *Check whether the other local parametrizations of $\mathbb{V}(p)$ centered at $(0, 1)$ lead to formal Puiseux series solutions. In the affirmative case, predict the commen denominator in the set of exponents. What about the local parametrizations of $\mathbb{V}(p)$ centered at $(\alpha, 0)$ where $\alpha^6 + 3\alpha^4 - \alpha^2 + 1 = 0$?*

There are more general type of local solutions than formal Puiseux series such as the zero $u(x) = c + \log(x)$ of $p = xu' - 1$. Such solutions are covered by so-called logarithmic transseries [13]. For a given differential polynomial p, we can derive the *Newton degree* of p (see the course on the Newton polygon method) which leads to a lower bound of the number of zeros as follows. For algebraic equations, the Newton degree coincides with the degree of p.

Theorem 48 (Theorem 1 in [12])**.** *Let $p \in \mathbb{K}[x]\{u\}$ be a differential polynomial of Newton-degree n. Then there are at least n logarithmic transseries solutions of $p = 0$.*

Example 49. *Consider the differential system $A = \{xu' - cu - x, c'\}$. Try to find its formal power series solutions. Are there other type of solutions?*

Using rational first integrals. In the general case, there is no algorithm for finding rational / formal power series solutions of realizations. If there are no solutions found, it can be the case that an algebraic invariance can be derived. More precisely, we define a *(rational) first integral* of (1) as any (rational) function W that satisfies $D(W) = 0$ with

$$D = \sum_j F_j \cdot \frac{\partial}{\partial t_j}.$$

Assume that a solution $\mathbf{t}(x)$ of (1) fulfills an algebraic relation $W(\mathbf{t}(x)) = 0$ for some irreducible W. Then also $\frac{d}{dx}(W(\mathbf{t}(x))) = 0$. We thus get

$$Q \cdot \sum_j \tfrac{\partial W}{\partial t_j}(\mathbf{t}(x)) \cdot t_j'(x) = Q \cdot \sum_j \tfrac{\partial W}{\partial t_j}(\mathbf{t}(x)) \cdot f_j = D(W(\mathbf{t}(x))),$$

and if $\mathbf{t}(x)$ is non-constant, $D(W)$ is in the ideal generated by W itself. W with this property is called *invariant algebraic hypersurfaces.* The special case where $D(W) = 0$ with $W \in \mathbb{K}(T)$, namely the rational first integrals, are of particular interest. Let $W = w_n/w_d$ be a rational first integral. Consider

$$L = \operatorname{numer}((w_n - c\, w_d) \circ (\mathcal{P}^{-1}(\mathbf{z}))). \tag{4}$$

Then we can add L to the given input-output equations A and compute a regular differential chain. Since L is not in the differential ideal generated by A, the new subsystems simplify. We note that not all of the solutions of A and $A \cup \{L\}$ coincide, but the general solution of A is preserved.

Example 50. *Let $p = (u')^2 u'' - u$. The corresponding algebraic variety $\mathbb{V}(p)$ has the rational parametrization $\mathcal{P} = (t_1, t_2, t_1/t_2^2)$ with the inverse $\mathcal{P}^{-1}(z_0, z_1, z_2) = (z_0, z_1)$. This leads to the realization*

$$t_1' = t_2, t_2' = t_1/t_2^2, u = t_1.$$

By making an ansatz and solving for the coefficients, we can find the rational first integral $W = t_2^4 - 2t_1^2$. This leads to $L = u'^4 - 2u^2 - c$.

The process described here can be seen as literally integrating the given system of differential equations. This leads to the parameter c which can be freely chosen such as it is the case for any indefinite integration. In the previous example, $\frac{d}{dx}L = p$ and we can replace p by L. This is exactly the case when using differential elimination applied to $\{p, L\}$.

Example 51. *The algebraic variety implicitly defined by the differential system*

$$\{p_1 = u_1'u_2' + 2u_1 - u_1' - 2u_2' + 2,\ p_2 = -u_1'u_2 - {u_2'}^2 + u_1^2 - u_1 + 4u_2 + u_2'\}$$

admits the rational parametrization

$$\mathcal{P} = \left(t_1, \frac{2(t_2 - t_1 - 1)}{t_2 - 1}, \frac{(t_2 - 1)(t_2 - t_1)}{2}, t_2\right).$$

The corresponding realization

$$\left\{t_1' = \frac{2(t_2 - t_1 - 1)}{t_2 - 1}, t_2' = 2, u_1 = t_1, u_2 = \frac{(t_2 - 1)(t_2 - t_1)}{2}\right\}$$

has the rational first integral

$$W(t_1, t_2) = -\frac{t_2^2 - 2t_1t_2 - 2t_2 + 2t_1}{2}.$$

We deduce

$$L = -u_2'^2 + 2u_1u_2' + 2u_2' - 2u_1 - 2c.$$

Computing the Thomas decomposition of $\{p, L\}$*, leading to an elimination of* u_1'*, gives in particular the simplified system*

$$\begin{aligned}\mathcal{F} = \{q_1 &= 2u_1u_2' - u_2'^2 - 2c - 2u_1 + 2u_2',\\ q_2 &= 3u_2'^3 - 4cu_1 - 2cu_2' - 8u_1u_2' - 8u_2'u_2' - 3u_2'^2 + 6c + 2u_1 + 8u_2 - 2u_2',\\ q_3 &= 2cu_1^2 - 2cu_2'^2 + 4u_1^2u_2 - u_2u_2'^2 - 2cu_1 + 2cu_2 + 2cu_2' - u_1^2 + 2u_2u_2' + u_2'^2 + u_1 - 2u_2 - u_2'\}.\end{aligned}$$

Exercise 52. *In Example 51:*

(1) *Could we have eliminated* u_2' *in* $\{p, L\}$ *instead of* u_1'*? What is a necessary criteria for that?*
(2) *Try to find a zero of* $\{p_1, p_2\}$ *and of* $\mathcal{F}$ *by using software. Is the zero* $\mathbf{u}(x) = (\frac{c+x^2}{x}, x^2 - c + x)$ *of* $\mathcal{F}$ *also a zero of* $\{p_1, p_2\}$*?*
(3) *Can we use the above approach to further simplify* $\mathcal{F}$*?*

Exercise 53. *Use the above approach to simplify the system given by*

$$\begin{aligned}p_1 &= u_2^2 - u_2',\\ p_2 &= 9u_1^2u_1'u_2' - 8u_1^2u_2' - 2u_1u_1'u_2 - u_1u_1'u_2' + u_1'^2u_2 - 2u_1'^2u_2' + 4u_1u_2 + 2u_1u_2' + u_1'^2\\ &\quad - 4u_1'u_2 + 8u_1'u_2' - 4u_1' + 4u_2 - 8u_2' + 4.\end{aligned}$$

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