The Fundamental Theorem of Tropical Differential Algebra over nontrivially valued fields

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Summary of contents

- Preliminaries and statement of the theorem;

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- Some definitions and sketch of proof;
- (Maybe) Tropical methods for radius of convergence of solutions to nonarchimedean differential equations.

Preliminaries: differential polynomials

(R, d_R) differential ring. Let

$$R\{x_1, \ldots, x_n\} := R[x_i^{(j)} \mid i = 1, \ldots, n; j \in \mathbb{N}]$$

Equipped with the differential $d(x_i^{(j)}) = x_i^{(j+1)}$ extending d_R, it is a differential ring.

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An element $r = (r_1, \ldots, r_n) \in R^n$ is a solution for $F \in R\{x_1, \ldots, x_n\}$ iff

$$F\big|_{x_i^{(j)} = d_R^j r_i} = 0$$

Preliminaries: semirings

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Example

- $\mathbb{T} := (\mathbb{R} \cup \{\infty\}, \min, +)$, *the tropical idempotent semiring;*
- *For* $n \in \mathbb{N}$, *let* $(\mathbb{T}_n, \oplus, \odot) := (\mathbb{R}^n \cup \{\infty\}, \min_{lex}, +)$. *It is an idempotent semiring. For* $n = 1$, *we recover the usual tropical semiring* $\mathbb{T}$.

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Let R be a ring, a *rank n valuation* is a map $v : R \to \mathbb{T}_n$ such that:

1. $v(0) = \infty$, $v(1) = v(-1) = 0$;
2. $v(ab) = v(a) \odot v(b)$;
3. $v(a + b) \oplus v(a) \oplus v(b)$ tropically vanishes (i.e. min is attained at least twice).

Preliminaries: setting

Let K be an uncountable, algebraically closed field of characteristic 0 equipped with a valuation $v_K : K \to \mathbb{T}$.

Let $(K[\![t]\!], d)$ be the differential ring of power series over K and $v : K[\![t]\!] \to \mathbb{T}_2$ the rank 2 valuation defined as:

$$a_{n_0} t^{n_0} + \ldots \longmapsto (n_0, v_K(a_{n_0})).$$

Preliminaries: tropicalization of differential polynomials

Given a differential polynomial $P \in K[\![t]\!]\{x_1, \ldots, x_n\}$ we define its tropicalization $\mathrm{trop}_v(P)$ with respect to v as the element of $\mathbb{T}_2\{x_1, \ldots, x_n\}$ obtained by applying v to the coefficients of P.

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Example

Let $P = 12t^2xx' + (-9 + 3t)x'' \in \mathbb{Q}_3[\![t]\!]\{x\}$*, then its tropicalization is:*

$$\text{trop}_v(P) = (2, 1)xx' + (0, 2)x''$$

Preliminaries: tropicalization of solutions

The idempotent semiring $\mathbb{T}[\![t]\!]$ can be endowed with the tropical differential:

$$d_{v_K}(t^n) = \begin{cases} v_K(n)t^{n-1} & n \geq 1 \\ \infty & n = 0. \end{cases}$$

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This is an additive map such that, for every $A, B \in \mathbb{T}[\![t]\!]$ the expression

$$d_{v_K}(AB) \oplus Bd_{v_K}(A) \oplus Ad_{v_K}(B)$$

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tropically vanishes. We denote $(\mathbb{T}[\![t]\!], d_{v_K})$ as $\mathbb{T}[\![t]\!]_{v_K}$.

Preliminaries: tropicalization of solutions

We tropicalize elements of $K[\![t]\!]$ via the map $\widetilde{v} : K[\![t]\!] \to \mathbb{T}[\![t]\!]_{v_K}$ applying v_K coefficientwise:

$$\sum_{i=0}^{\infty} a_i t^i \mapsto \sum_{i=0}^{\infty} v_K(a_i) t^i$$

This maps commutes with the differentials.

Applying $\widetilde{v}$ coordinatewise we obtain the tropicalization map $\mathrm{trop}_{\widetilde{v}} : K[\![t]\!]^n \to \mathbb{T}[\![t]\!]^n_{v_K}$.

Preliminaries: tropical solutions

Let $\Phi : \mathbb{T}[\![t]\!]_{v_K} \to \mathbb{T}_2$ be the homomorphism of semirings

$$b_{n_0} t^{n_0} + \cdots \mapsto (n_0, b_{n_0}).$$

Given a $P \in K[\![t]\!]\{x_1, \ldots, x_n\}$ and $S = (S_1, \ldots, S_n) \in \mathbb{T}[\![t]\!]^n_{v_K}$, we say that S is a solution for the tropicalization of P if when plugging $\Phi(d^j S_i)$ for $x_i^{(j)}$ in $\mathrm{trop}_v(P)$ the result tropically vanishes in $\mathbb{T}_2$.

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Example

Let P as before, $\mathrm{trop}_v(P) = (2,1)xx' + (0,2)x''$ *and* $S = 0 + 1t + (-1)t^4 \in \mathbb{T}[\![t]\!]_{v_3}$*, then S is a solution for* $\mathrm{trop}_v(P)$*:*

$$\begin{aligned}\mathrm{trop}_v P(S) &= (2,1) \odot \Phi(S) \odot \Phi(dS) \oplus (0,2) \odot \Phi(d^2 S) = \\ &= (2,1) \odot (0,0) \odot (0,1) \oplus (0,2) \odot (2,0) = \\ &= (2,2) \oplus (2,2)\end{aligned}$$

.

Fundamental theorem of tropical differential algebra

Theorem

Let K be an uncountable algebraically closed field of characteristic 0 and $v_K : K \to \mathbb{T}$ a valuation. Let I be a differential ideal in $K[\![t]\!]\{x_1, \ldots, x_n\}$, then the following equality holds:

$$\mathrm{Sol}_{\mathbb{T}[\![t]\!]_{v_K}}(\mathrm{trop}_v(I)) = \mathrm{trop}_{\widetilde{v}}(\mathrm{Sol}_{K[\![t]\!]}(I)).$$

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- The Fundamental Theorem had already been proven by Aroca, Garay, Toghani in the case of trivial valuation $\rightsquigarrow$ only tropical info about the *support* of power series solutions.
- In the nontrivially valued case we want to have a *valuated* version of the fundamental theorem $\rightsquigarrow$ tropical info about *convergence* of power series solutions.
- For *p*-adic differential equations, the convergence radius function of solutions is a piecewise linear function in the norm of the expansion point. We want to have tropical methods for computing it.

The polynomials $F_{I,r}$

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By results of Ritt, there is a finite number of elements $f_1, \ldots, f_s \in I$ such that

$$\mathrm{Sol}_{K[\![t]\!]}(I) = \bigcap_{l=1}^{s} \mathrm{Sol}_{K[\![t]\!]}(f_l)$$

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$$\mathrm{Sol}_{K[\![t]\!]}(I) = \bigcap_{l=1}^{s} \mathrm{Sol}_{K[\![t]\!]}(f_l)$$

For all $l = 1, \ldots, s$, $r \in \mathbb{N}$, set

$$F_{l,r} := (d^r f_l)|_{t=0} \in K[x_i^{(j)} \mid i = 1, \ldots, n; j \in \mathbb{N}]$$

and

$$A_\infty := V\left(\{F_{l,r}\}_{\substack{1 \le l \le s \\ r \in \mathbb{N}}}\right) \subset \left(K^{\mathbb{N}}\right)^n .$$

The function Ψ

The map $\Psi : K^{\mathbb{N}} \to K[\![t]\!]$ defined as

$$(a_j)_{j\in\mathbb{N}} \mapsto \sum_{j=0}^{\infty} \frac{1}{j!} a_j t^j$$

is a bijection. We denote the bijection $\left(K^{\mathbb{N}}\right)^n \to K[\![t]\!]^n$ again by Ψ.

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Given $f \in K[\![t]\!]\{x_1, \ldots, x_n\}$ and $a \in \left(K^{\mathbb{N}}\right)^n$ the following equality holds:

$$f\left(\Psi(a)\right) = \sum_{r=0}^{\infty} \left(\frac{1}{r!} \left(d^r(f)\right)|_{t=0}(a) \right) t^r.$$

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Thus:

$$\mathrm{Sol}_{K[\![t]\!]}(I) = \Psi(A_\infty).$$

Truncations

For $m \in \mathbb{N}$, let N_m be the smallest natural number such that

$$F_{l,r} \in K[x_i^{(j)} \mid i = 1, \ldots, n; j \leq N_m] \quad \text{for all } 1 \leq l \leq s,\ 0 \leq r \leq m$$

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$$A_\infty = \varprojlim A_m.$$

Fibers of tropicalization

Let $m \in \mathbb{N}$ and $S := (S_1, \ldots, S_n) \in \mathbb{T}[\![t]\!]^n_{v_K}$, where we write S_i as $\sum c_{i,j} t^j$ for every $i = 1, \ldots, n$. With this notation, we define:

$$(\mathbb{V}_\infty)^{v_K}_S := v_K^{-1}\left((c_{i,j} + v_K(j!))_{\substack{i=1,\ldots,n \\ j \in \mathbb{N}}} \right) \in \left(K^{\mathbb{N}}\right)^n$$

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Furthermore, let

$$(A_m)_S^{v_K} := A_m \cap (\mathbb{V}_m)_S^{v_K}. \qquad (A_\infty)_S^{v_K} := A_\infty \cap (\mathbb{V}_\infty)_S^{v_K}$$

Criterion for lifting tropical solutions

As before for any $S \in \mathbb{T}[\![t]\!]^n_{v_K}$ we have $(A_\infty)^{v_K}_S = \varprojlim (A_m)^{v_K}_S$.

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Remark

$$S \in \mathrm{trop}_{\tilde{v}}(\mathrm{Sol}_{K[\![t]\!]}(I)) \iff (A_\infty)_S^{v_K} \neq \emptyset.$$

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Proposition

$(A_\infty)_S^{v_K} \neq \emptyset \iff (A_m)_S^{v_K} \neq \emptyset$ for all $m \in \mathbb{N}$.

Proof of the theorem

Let $\Psi_{\text{trop}} : \mathbb{T}^{\mathbb{N}} \to \mathbb{T}[\![t]\!]_{v_K}$ be the bijective map defined by:

$$\Psi_{\text{trop}}((b_j)_{j\in\mathbb{N}}) = \sum_{j=0}^{\infty}(b_j - v_K(j!))t^j.$$

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Its inverse is defined as follows:

$$\Psi_{\text{trop}}^{-1}(S) = \left((d_{v_K}^j S)|_{t=\infty}\right)_{j\in\mathbb{N}}.$$

We denote again by Ψ_{trop} the map $(\mathbb{T}^{\mathbb{N}})^n \to \mathbb{T}[\![t]\!]_{v_K}^n$ obtained by applying Ψ_{trop} coordinatewise.

Proof of the theorem

Lemma

For every $m \in \mathbb{N}$*, let* $\pi_m : (\mathbb{T}^{\mathbb{N}})^n \to (\mathbb{T}^{N_m+1})^n$ *be the projection sending every entry to its first* $N_m + 1$ *coordinates. The following inclusion holds for every* $m \in \mathbb{N}$*:*

$$\pi_m \circ \Psi_{\mathrm{trop}}^{-1}\left(\mathrm{Sol}_{\mathbb{T}[\![t]\!]_{v_K}}(\mathrm{trop}_v(I))\right) \subset V^{\mathrm{trop}}\left(\left\{\mathrm{trop}_{v_K}(F_{l,r})\right\}_{\substack{l=1,\dots,s \\ 0\leq r\leq m}}\right).$$

Proof of the theorem

- The inclusion $\mathrm{Sol}_{\mathbf{S}}(\mathrm{trop}_v(I)) \supseteq \mathrm{trop}_{\tilde{v}}(\mathrm{Sol}_{K[\![t]\!]}(I))$ follows from the properties of $v, \widetilde{v}$ and Φ;

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- For every $m \in \mathbb{N}$ by the Fundamental Theorem of Tropical Geometry, we have:

$$\mathrm{trop}_{v_K}(A_m) = V^{\mathrm{trop}}\left(\left\{\mathrm{trop}_{v_K}(F_{l,r})\right\}_{\substack{l=1,\dots,s\\0\leq r\leq m}}\right).$$

Let $S \in \mathrm{Sol}_{\mathbb{T}[\![t]\!]_{v_K}}(\mathrm{trop}_v(I))$. From the Lemma above, we have, for all $m \in \mathbb{N}$:

$$\pi_m \circ \Psi_{\mathrm{trop}}^{-1}(S) \in V^{\mathrm{trop}}\left(\left\{\mathrm{trop}_{v_K}(F_{l,r})\right\}_{\substack{l=1,\dots,n\\0\leq r\leq m}}\right) = \mathrm{trop}_{v_K}(A_m)$$

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Let $S \in \mathrm{Sol}_{\mathbb{T}[\![t]\!]_{v_K}}(\mathrm{trop}_v(I))$. From the Lemma above, we have, for all $m \in \mathbb{N}$:

$$\pi_m \circ \Psi^{-1}_{\mathrm{trop}}(S) \in V^{\mathrm{trop}}\left(\left\{\mathrm{trop}_{v_K}(F_{l,r})\right\}_{\substack{l=1,\dots,n\\ 0\le r\le m}}\right) = \mathrm{trop}_{v_K}(A_m)$$

$\implies$ for all $m \in \mathbb{N}$ there exists an element $x \in A_m$ whose tropicalization is $\pi_m \circ \Psi^{-1}_{\mathrm{trop}}(S)$, i.e. $x \in (A_m)^{v_K}_S$. Finally, thanks to Proposition and Remark,

$$(A_m)^{v_K}_S \neq \emptyset \text{ for all } m \in \mathbb{N} \iff S \in \mathrm{trop}_{\tilde{v}}(\mathrm{Sol}_{K[\![t]\!]}(I)).$$

Last remark on the theorem

There is a way to define the initial of a differential ideal I with respect to a weight vector $S \in \mathbb{T}[\![t]\!]^n_{v_K}$, so we get the following:

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Theorem

Under the same hypothesis of the previous theorem, the following equalities hold:

$$\mathrm{trop}_{\tilde{v}}(\mathrm{Sol}_{K[\![t]\!]}(I)) = \mathrm{Sol}_{\mathbb{T}[\![t]\!]_{v_K}}(\mathrm{trop}_v(I)) = \{S \in \mathbb{T}[\![t]\!]_{v_K}^n \mid In_S(I) \textit{ does not contain a monomial}\}$$

Radius of convergence

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$$|n|_p := p^{-v_p(n)}$$

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Given $A = \sum_{i=0}^{\infty} a_i t^i \in \mathbb{C}_p[\![t]\!]$, its radius of convergence is

$$r(A) := \sup\{r \in [0, \infty) \mid \lim_{i \to +\infty} |a_i|_p r^i = 0\} \in [0, \infty].$$

Radius of convergence

Given $B = \sum_{i=0}^{n} b_i t^i \in \mathbb{T}[\![t]\!]$, we define its p-adic radius of convergence as:

$$r_p(B) := \sup\{r \in [0, \infty) \mid \lim_{i \to +\infty} p^{-b_i} r^i = 0\} \in [0, \infty]$$

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Corollary

Let $I \in K[\![t]\!]\{x\}$. Then:

$$\{r(A) \mid A \in \mathrm{Sol}_{K[\![t]\!]}(I)\} = \{r_p(S) \mid S \in \mathrm{Sol}_{\mathbb{T}[\![t]\!]_{v_K}}(\mathrm{trop}_v(I))\}.$$

Radius of convergence

Given an ideal $J \in K[x_1, \ldots, x_n]$ a tropical basis for J is a subset G of I such that

$$V^{\text{trop}}(\text{trop}_v(I)) = \bigcap_{g \in G} V^{\text{trop}}(\text{trop}_v(g))$$

In general this set is larger than a set of generators for J, and it depends on some matroidal conditions.

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Analogously, Fink-Toghani proposed a notion of tropical differential basis for a differential ideal $I \subset K[\![t]\!]\{x_1, \ldots, x_n\}$: a subset G of I such that (dropping some notation)

$$\text{Sol}(\text{trop}_v(I)) = \bigcap_{g \in G} \bigcap_{r=0}^{\infty} \text{Sol}(\text{trop}_v(d^r g))$$

and they show an (easy) example where such a basis is not finite.

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To make these methods useful for computations:

- criterion to verify if a subset of I is a tropical differential basis?
- criterion that certifies that a given ideal admits a finite tropical differential basis?
- a different notion of tropical differential basis, that is ensured to be always finite?

Radius of convergence

To make these methods useful for computations:

- criterion to verify if a subset of I is a tropical differential basis?
- criterion that certifies that a given ideal admits a finite tropical differential basis?
- a different notion of tropical differential basis, that is ensured to be always finite?

Thank you!

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