CTLNs from a Tropical Viewpoint

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June 19, 2023

From Graphs to Differential Equations

Given a simple directed graph G with n vertices we can define a matrix $W = (w_{ij})$ as follows:

If $i = j$, then $w_{ij} = 0$.

If $i < -j$ in G, then $w_{ij} = -1 + \epsilon$.

Else $w_{ij} = -1 - \delta$.

Following Curto and Morrison [3], a Combinatorial Linear Threshold Network (CTLN) is given by the differential equations

$$\frac{dx}{dt} = -x + [Wx + \theta]_+$$

where $\theta \in \mathbb{R}^n$ is a vector of constants, and $[v]_+$ denotes the vector which coordinates are the maximum between the coordinates of v and 0. We additionally require that $\delta > 0$ and $0 < \epsilon < \frac{\delta}{1+\delta}$.

First example

Consider the digraph G with two nodes A and B and two edges AB and BA.

It has associated the CTLN

$$\frac{dx}{dt} = -x + Max[-0.75y + 1, 0]$$

$$\frac{dy}{dt} = -y + Max[-0.75x + 1, 0]$$

A typical method used in the study of Ordinary Differential Equations (ODE's) is the analysis of the **nullclines**.

Nullclines

Recall that the nullclines of a system of two ODEs

$$\frac{dx}{dt} = P(x, y)$$

$$\frac{dy}{dt} = Q(x, y)$$

is given as the two curves

$$\{(x, y) \in \mathbb{R}^2 \text{ such that } P(x, y) = 0\}$$

$$\{(x, y) \in \mathbb{R}^2 \text{ such that } Q(x, y) = 0\}$$

The nullclines of Ex. 1

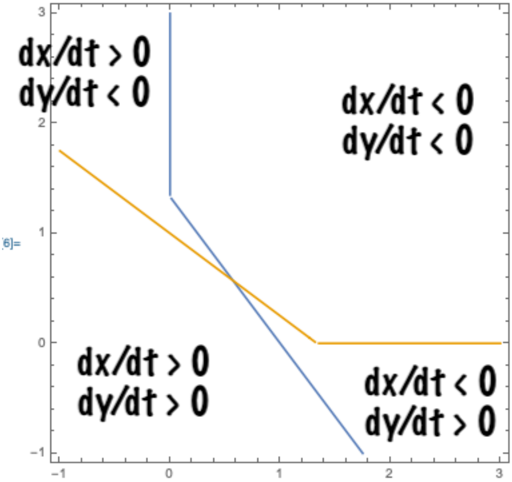
In our first example we have that the nullclines are the curves given by the system of equations

$$-x + Max[-0.75y + 1, 0] = 0$$

$$-y + Max[-0.75x + 1, 0] = 0$$

How to solve it?

The nullclines of Ex. 1



Equilibrium points

There is one equilibrium point.
Its coordinates are

$$(x_{12}, y_{12}) = (4/7, 4/7)$$

What kind of equilibrium point is it?
Observe that it belongs to the region

$$-0.75y + 1 > 0, \quad -0.75x + 1 > 0$$

so near this point the differential equations have the form

$$\frac{dx}{dt} = -x - 0.75y + 1$$

$$\frac{dy}{dt} = -0.75x - y + 1$$

It is a node

The characteristic polynomial of the matrix

$$A = \begin{pmatrix} -1 & -0.75 \\ -0.75 & -1 \end{pmatrix}$$

is $p(\lambda) = (-1-\lambda)^2 - \frac{9}{16}$
The eigenvalues of A are both negative

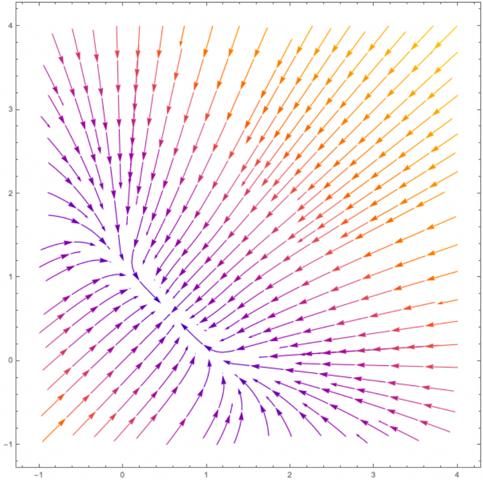
$$-1 \pm \frac{3}{4}$$

The equilibrium point is a sink, and is asymptotically stable.

The phase plane

```
In[3]:= StreamPlot[{p, q}, {x, -1, 4}, {y, -1, 4}]
        representación de flujo
```

Out[3]=



Some solutions

```
In[11]:= Plot[Evaluate[{x[t], y[t]} /. sol], {t, 0, 30}]
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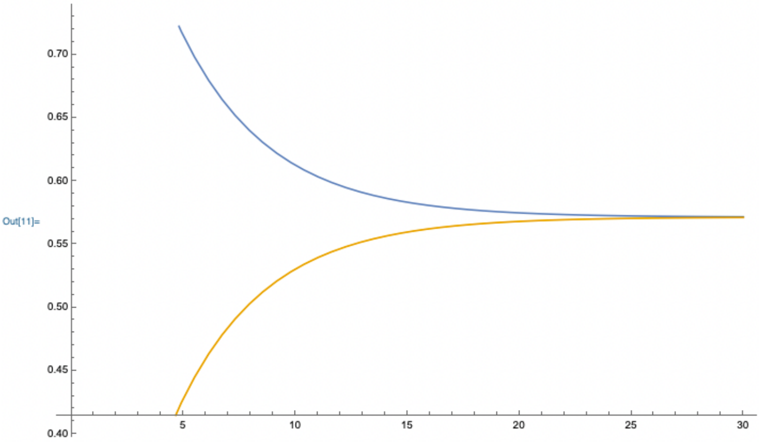


Arrangement of hyperplanes

Recall that the differential equations of this example are

$$\frac{dx}{dt} = -x + Max[-0.75y + 1, 0]$$

$$\frac{dy}{dt} = -y + Max[-0.75x + 1, 0]$$

Define $F(x, y) = -0.75y + 1$ and $G(x, y) = -0.75x + 1$. Observe that the equations $F(x, y) = 0$ and $G(x, y) = 0$ define an arrangement of two lines that divide the plane into four regions given by the inequalities

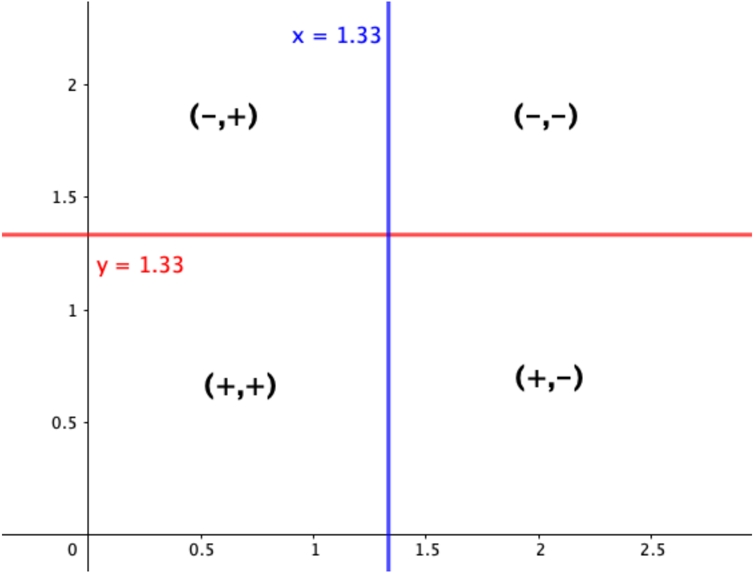
$$F(x, y) > 0, \quad G(x, y) > 0$$

$$F(x, y) > 0, \quad G(x, y) < 0$$

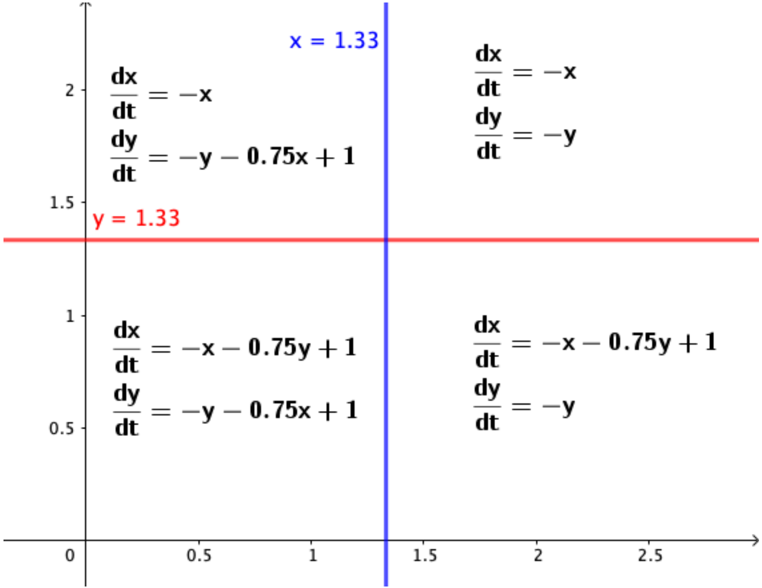
$$F(x, y) < 0, \quad G(x, y) > 0$$

$$F(x, y) < 0, \quad G(x, y) < 0$$

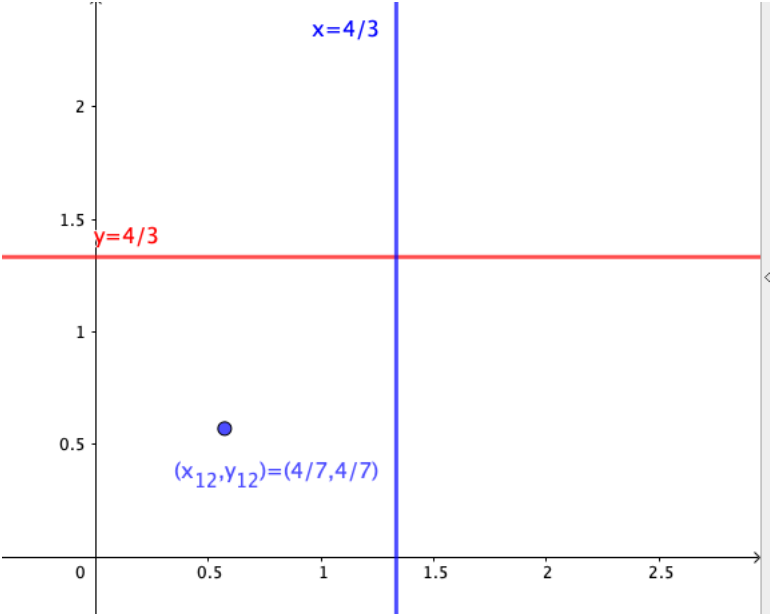
Arrangement of hyperplanes



Arrangement of hyperplanes



Arrangement of hyperplanes



CTLN as a patchwork of linear systems

1. The arrangement of hyperplanes divides $\mathbb{R}^n$ in chambers.
2. Restricted to each chamber, the differential equations become linear.
3. Each linear system has his own equilibrium point.
4. This point can be inside or outside of the chamber.
5. If the equilibrium point of the linear system belongs to the corresponding chamber, it is an equilibrium point of the non linear system CTLN.

We can label each chamber with a subset of $[n]$.
The number of fixed points of a CTLN with n neurons is at most $2^n - 1$.

Second example

Consider the digraph with two nodes A and B with no edges between A and B.

Second example

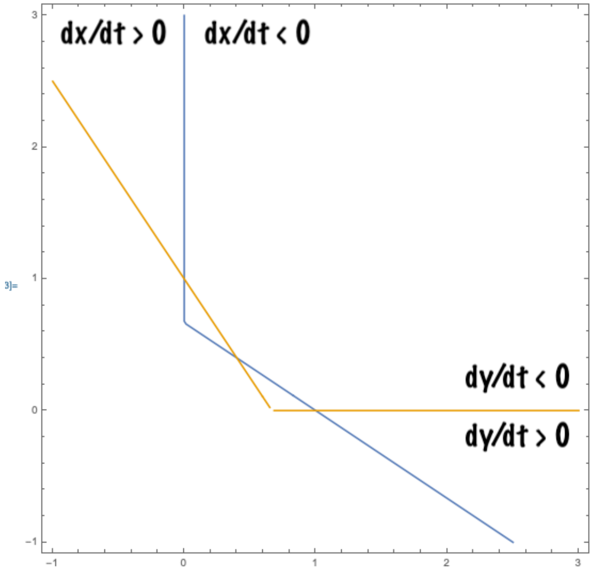
It has associated the CTLN

$$\frac{dx}{dt} = -x + Max[-1.5y + 1, 0]$$

$$\frac{dy}{dt} = -y + Max[-1.5x + 1, 0]$$

How are its nullclines?

The nullclines of Ex. 2



Equilibrium points

There are three equilibrium points.
Their coordinates are

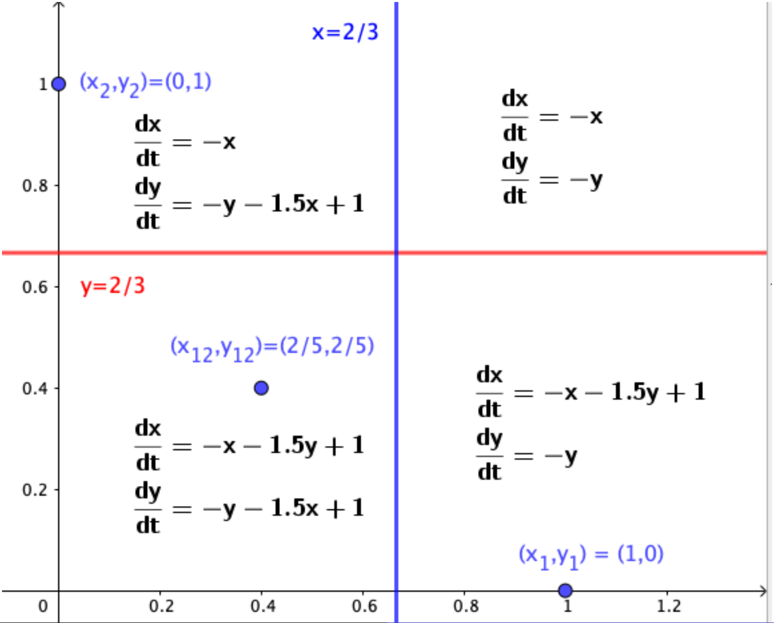
$$(x_1, y_1) = (1, 0)$$

$$(x_2, y_2) = (0, 1)$$

$$(x_{12}, y_{12}) = (2/5, 2/5)$$

What kind of equilibrium point is each one?

Remember: each equilibrium point belongs to a different chamber. And on each chamber the CTLN restricts to a different system of linear differential equations.


x=2/3
1
(x2,y2)=(0,1)
dx/dt = −x
dy/dt = −y − 1.5x + 1
dx/dt = −x
dy/dt = −y
0.8
0.6
y=2/3
(x12,y12)=(2/5,2/5)
0.4
dx/dt = −x − 1.5y + 1
dy/dt = −y
dx/dt = −x − 1.5y + 1
dy/dt = −y − 1.5x + 1
0.2
(x1,y1) = (1,0)
0
0.2
0.4
0.6
0.8
1
1.2

The point (x_1, y_1).

Its coordinates are $(1, 0)$, and the equations are

$$\frac{dx}{dt} = -x - 1.5y + 1$$

$$\frac{dy}{dt} = -y$$

The matrix is

$$A = \begin{pmatrix} -1 & -1.5 \\ 0 & -1 \end{pmatrix}$$

The point is a **sink**.

The point (x_2, y_2).

Its coordinates are $(0, 1)$, and the equations are

$$\frac{dx}{dt} = -x$$

$$\frac{dy}{dt} = -1.5x - y + 1$$

The matrix is

$$A = \begin{pmatrix} -1 & 0 \\ -1.5 & -1 \end{pmatrix}$$

The point is a **sink**.

The point (x_{12}, y_{12}).

Its coordinates are $(2/5, 2/5)$, and the equations are

$$\frac{dx}{dt} = -x - 1.5y + 1$$

$$\frac{dy}{dt} = -1.5x - y - 1$$

The matrix is

$$A = \begin{pmatrix} -1 & -1.5 \\ -1.5 & -1 \end{pmatrix}$$

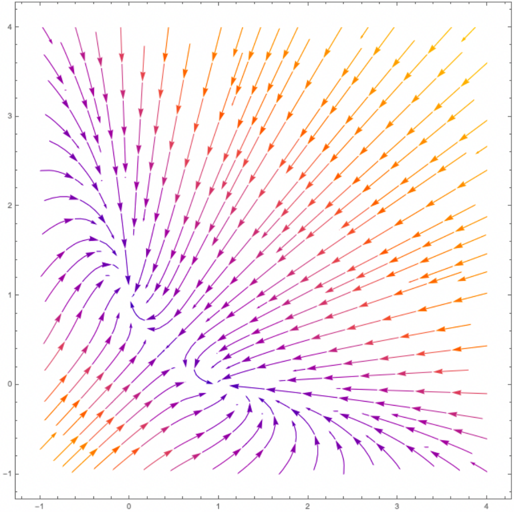
The eigenvalues are -1 ± 1.5.

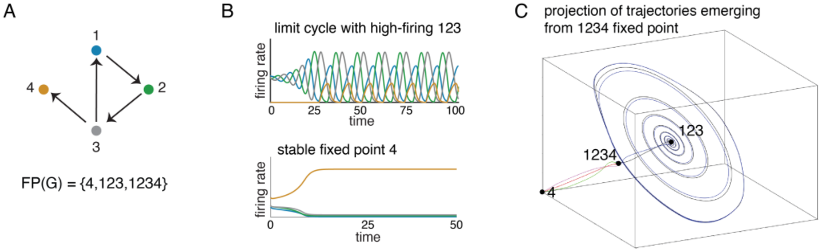
The point is a **saddle**.

The phase plane

In[6]:= StreamPlot[{p, q}, {x, -1, 4}, {y, -1, 4}]
representación de flujo

Out[6]=



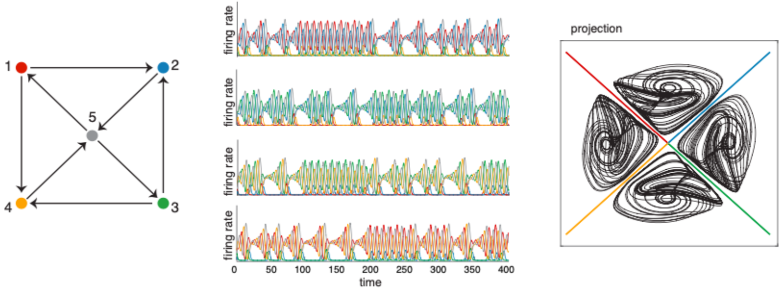


Numerically, we observe limit cycles ...

What about the theory?

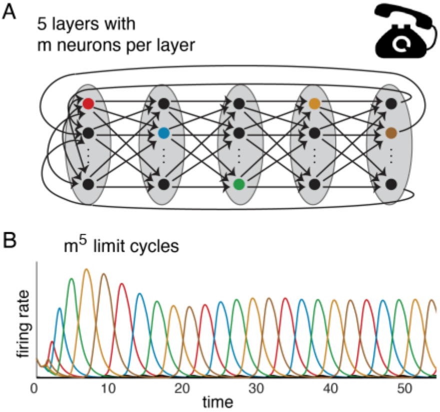
Can tropical geometry help us?

Fourth example



Is it chaos or only looks like chaos?

Fifth example



How are the cell decompositions of $\mathbb{R}^n$ associated to this system?
What is the relation between the blocks and the whole system?

Fixed points set

Let $p = (p_1, p_2, \ldots, p_n)$ be a fixed point of a CTLN on n neurons.
The **support** of p is the subset of $i \in \{1, 2, \ldots, n\}$ s. t. $p_i \neq 0$.
In the second example

$$supp(1, 0) = \{1\}$$

$$supp(0, 1) = \{2\}$$

$$supp(2/5, 2/5) = \{1, 2\}$$

Let FP be set of all supports of all fixed points of a CTLN.
In the second example

$$FP = \{1, 2, 12\}$$

Problem 1

Given a digraph with n vertices and the corresponding CTLN.

Count $|FP|$, the number of fixed points of the CTLN.

In other words:
Count the number of solutions of the system of equations given by:

$$-x_i + Max\{\sum w_{ji}x_j + 1, 0\} = 0, \quad i = 1, 2, \dots, n$$

Necessarily:

1. $1 \leq |FP| \leq 2^n - 1.$
2. $|FP|$ is odd.

Following Brugallé and Shaw [1], we recall the definition of a tropical curve.

Tropical curve

Let $P(x,y) = " \sum_{i,j} a_{i,j}x^i y^j " = max\{a_{i,j} + ix + jy\}$ be a tropical polynomial. The tropical curve C defined by $P(x,y)$ is the set of points $(x,y) \in \mathbb{R}^2$ such that there exists pairs $(i,j) \neq (k,l)$ satisfying

$$a_{i,j} + ix + jy = a_{k,l} + kx + ly$$

The weight of an edge e of C is the GCD of the numbers $|i-k|$ and $|j-l|$ which correspond to this edge.

Bézout's theorem

Two algebraic curves in the plane, of degrees d_1 and d_2 respectively, intersect in $d_1 d_2$ points.

Is this true for tropical curves?

Yes, but counting multiplicities.

Multiplicity

Tropical multiplicity

Let C_1 and C_2 be two tropical curves which intersect in a finite number of points and away from the vertices of the two curves. If p is a point of intersection of C_1 and C_2, the tropical multiplicity of p as an intersection of C_1 and C_2 is the area of the parallelogram dual to p in the dual subdivision of $C_1 \cup C_2$.

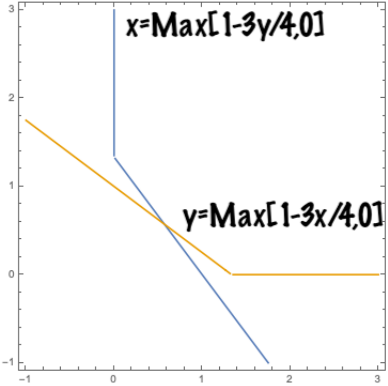
Tropical Bézout's theorem

Theorem (Sturmfels)

Let C_1 and C_2 be two tropical curves of degrees d_1 and d_2 respectively, intersecting in a finite number of points away from the vertices of the two curves. Then the sum of the tropical multiplicities of all points in the intersection of C_1 and C_2 is equal to $d_1 d_2$.

Help!

Return to the nullclines of example 1.



They are not tropical curves!
What can I do?

Tropical closure

Benoît Bertrand suggested me to consider its tropical closure. Look at the y-nullcline. It is given by the equation

$$y = Max\{0, 1 - 3x/4\}$$

To avoid working with non integer coefficients, multiply by 4

$$4y = Max\{0, 4 - 3x\}$$

Add $3x$ to the three quantities

$$3x + 4y = Max\{3x, 4\}$$

Now consider the set of points in $\mathbb{R}^2$ such that two of the three quantities tie

$$3x + 4y, 3x, 4$$

This is the tropical curve associated to the tropical polynomial

$$"x^3y^4 + x^3 + 4"$$

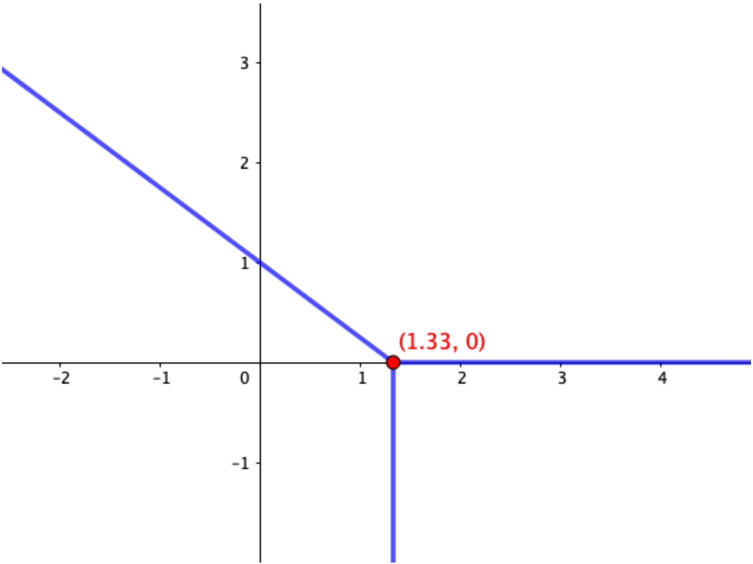
Play the game!

Lucía López de Medrano taught us how to associate a tropical curve with a game where we are looking for two winners of the three players:

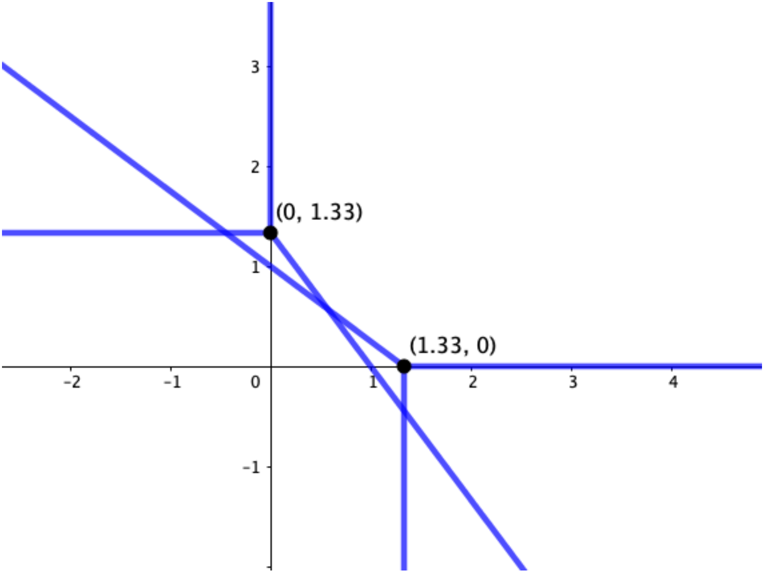
1. If $3x + 4y = 3x$, then $y = 0$.
2. If $3x + 4y = 4$, then $y = 1 - 3x/4$.
3. If $3x = 4$, then $x = 4/3$.

We find a tropical curve that contains the y-nullcline!

Plot that curve!

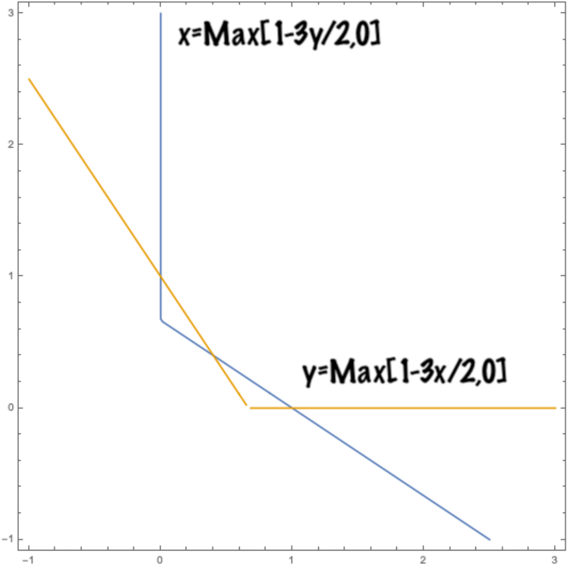


Plot the other one!



The second example

We can do the same with the nullclines of the second example.



Look at the y-nullcline. It is given by the equation

$$y = Max\{0, 1 - 3x/2\}$$

To avoid working with non integer coefficients, multiply by 2

$$2y = Max\{0, 2 - 3x\}$$

Add $3x$ to the three quantities

$$3x + 2y = Max\{3x, 2\}$$

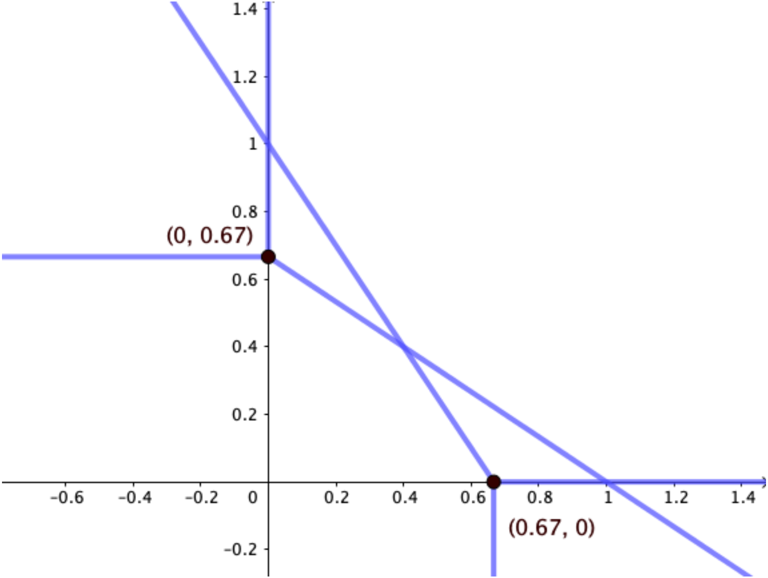
Now consider the set of points in $\mathbb{R}^2$ such that two of the three quantities tie

$$3x + 2y, 3x, 2$$

This is the tropical curve associated to the tropical polynomial

$$"x^3y^2 + x^3 + 2"$$

A second intersection of tropical curves



One more example

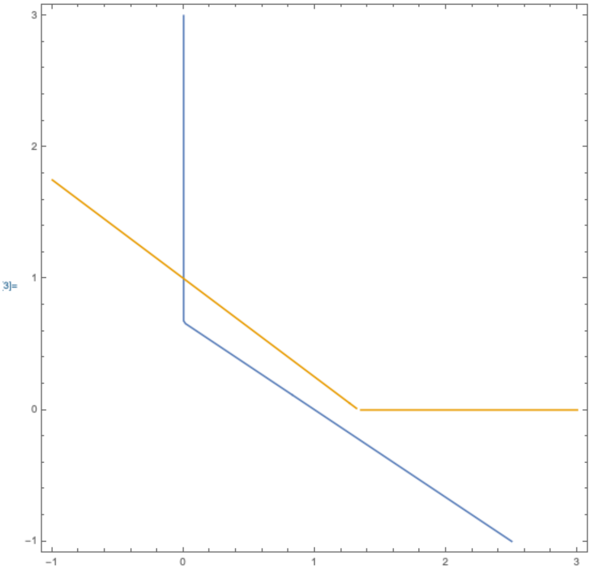
Consider the directed graph

The associated equations are

$$\frac{dx}{dt} = -x + Max[-1.5y + 1, 0]$$

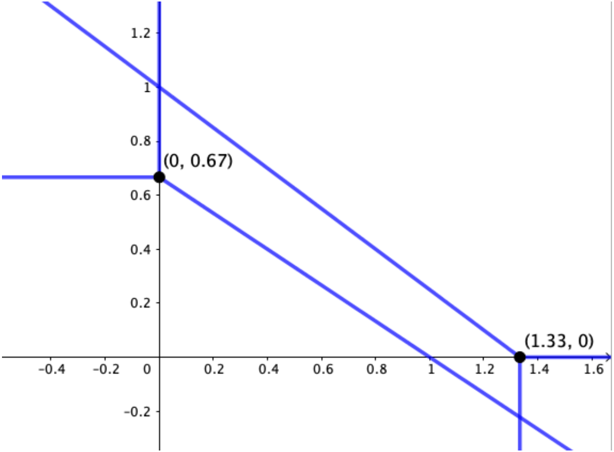
$$\frac{dy}{dt} = -y + Max[-0.75x + 1, 0]$$

The nullclines



The tropical curves

The associated polynomials are "$x^4y^3 + y^3 + 4$" and "$x^3y^2 + x^3 + 2$" and they look like this



Questions and work in progress

1. What about the multiplicities?
2. Understand and apply the Bernstein theorem and the Newton polytopes. See Sturmfels [4] and Katz [5].
3. Are there tropical analogues of the nerve theorems for CTLNs? See for example, Burtscher et al. [6].

To be continued ...

References

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A bit of tropical geometry.

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Predicting neural network dynamics via graphical analysis.

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Burtscher, Curto, Ebli, Egas, Morrison, Patania, Sanderson (2021).
Nerve theorems for fixed point of neural networks

C. Lienkaemper (2022)

Last but not least

THANK YOU!
¡GRACIAS!