Cluster structures and Tropicalization

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Overview

1. What is a cluster structure?
2. What is tropicalization?
3. How are they related?

What is a cluster structure?

For today: a projective variety X has a *cluster structure*, if there exist an embedding of X such that its homogeneous coordinate ring is a *cluster algebra*.

Examples: Grassmannians, (partial) flag varieties, Schubert varieties, (some) del Pezzo surfaces, ...

Cluster algebras

A *cluster algebra*[1] $A \subset \mathbb{C}(x_1, \dots, x_n)$ is a commutative ring defined recursively by

1. *seeds*: maximal sets of algebraically independent algebra generators,
   its elements are called *cluster variables*;
2. *mutation*: an operation to create a new seed from a given one by replacing one element.

For example, $s_0 = \{x_1, \dots, x_n\}$ then mutating at a variable x_k we get

$$\mu_k(s_0) = \{x_1, \dots, x_{k-1}, x_k', x_{k+1}, \dots, x_n\},$$

where $x_k x_k' = \mathbf{x}^{m_1} + \mathbf{x}^{m_2}$ and m_1, m_2 are encoded in some combinatorial data.

[1] Defined by Fomin–Zelevinsky.

Example: $\mathrm{Gr}_2(\mathbb{C}^4)$

$\mathrm{Gr}_2(\mathbb{C}^4) = \{V \subset \mathbb{C}^4 \mid \dim V = 2\}$ with Plücker embedding:

$$\begin{array}{rcl} \mathrm{Gr}_2(\mathbb{C}^4) & \hookrightarrow & \mathbb{P}(\wedge^2\mathbb{C}^4) \\ V = \langle v_1, v_2 \rangle & \mapsto & [v_1 \wedge v_2] \end{array}$$

its homogeneous coordinate ring

$$A_{2,4} = \frac{\mathbb{C}[p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34}]}{p_{13}p_{24} = p_{12}p_{34} + p_{14}p_{23}}$$

is a cluster algebra with two seeds:

$$\begin{array}{rcl} s_0 & = & \{p_{12}, p_{23}, p_{34}, p_{14}, p_{13}\}, \text{ and} \\ s_1 & = & \{p_{12}, p_{23}, p_{34}, p_{14}, p_{24}\}. \end{array}$$

Example: Grassmannains

More generally, let $A_{k,n}$ be the homogeneous coordinate ring of $\mathrm{Gr}_k(\mathbb{C}^n)$ with Plücker embedding.

Theorem (Scott)

$A_{k,n}$ is a cluster algebra.

$k \leq 2$ Plücker coordinates $=$ cluster variables,

$k \geq 3$ Plücker coordinates $\subsetneq$ cluster variables,

$k = 2$ or $k = 3$ and $n \in \{6, 7, 8\}$ finitely many seeds.

Example: $\mathcal{F}\ell_3$

$\mathcal{F}\ell_3 = \{\{0\} \subset V_1 \subset V_2 \subset \mathbb{C}^3 \mid \dim V_i = i\}$ with Plücker embedding:

$$\mathcal{F}\ell_3 \hookrightarrow \mathrm{Gr}_1(\mathbb{C}^3) \times \mathrm{Gr}_2(\mathbb{C}^3) \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2.$$

Its homogeneous coordinate ring

$$A_3 = \frac{\mathbb{C}[p_1, p_2, p_3, p_{12}, p_{13}, p_{23}]}{p_2 p_{13} = p_1 p_{23} + p_3 p_{12}}$$

is a cluster algebra with two seeds:

$$\begin{aligned} s_0 &= \{p_1, p_3, p_{12}, p_{23}, p_2\}, \text{ and} \\ s_1 &= \{p_1, p_3, p_{12}, p_{23}, p_{13}\}. \end{aligned}$$

Example: Flag varieties

More generally, let A_n be the homogeneous coordinate ring of $\mathcal{F}\ell_n$ with Plücker embedding.

Theorem (Berenstein-Fomin-Zelevinsky)

A_n is a cluster algebra.

$n = 3$ Plücker coordinates = cluster variables,

$n \geq 4$ Plücker coordinates $\subsetneq$ cluster variables,

$n \leq 6$ finitely many seeds

Cluster toric degenerations

For every seed s we get a *toric degeneration*[2] $\pi : \mathcal{X}_s \to \mathbb{A}^n$ with

$$\pi^{-1}(\mathbf{1}) = X \quad \text{and} \quad \pi^{-1}(0) = X_{s,0} \quad \text{toric variety.}$$

The mutation relations $x_k x_k' = \mathbf{x}^{m_1} + \mathbf{x}^{m_2}$ in X are deformed to $x_k x_k' = \mathbf{x}^{m_i}$ in $X_{s,0}$.

Example

For $\mathcal{F}\ell_3$ we have two such toric degenerations:

$$\begin{aligned} \mathcal{X}_{s_0} &: \langle p_2 p_{13} - p_1 p_{23} - t p_3 p_{12} \rangle \\ \mathcal{X}_{s_1} &: \langle p_2 p_{13} - t p_1 p_{23} - p_3 p_{12} \rangle \end{aligned}$$

[2] Due to Gross–Hacking–Keel–Kontsevich.

Gröbner degenerations

For $I \subset \mathbb{C}[x_1, \dots, x_n]$ an ideal, $f = \sum c_\alpha \mathbf{x}^\alpha \in I$ and $w \in \mathbb{R}^n$ we define the *initial form* of f with respect to w

$$\mathrm{in}_w(f) := \sum_{w\cdot\beta = \min_{c\alpha \neq 0}\{w\cdot\alpha\}} c_\beta \mathbf{x}^\beta.$$

The *initial ideal* of I wrt w is $\mathrm{in}_w(I) := \langle \mathrm{in}_w(f) \ : \ f \in I \rangle$.

For every w we have a *Gröbner degeneration* $\pi : \mathcal{V} \to \mathbb{A}^1$ with

$$\pi^{-1}(1) = V(I) \text{ and } \pi^{-1}(0) = V(\mathrm{in}_w(I)).$$

Aim: $\mathrm{in}_w(J)$ is *binomial and prime* $\Rightarrow V(\mathrm{in}_w(I))$ is toric.

Tropicalization

Definition

For $I \subset \mathbb{C}[x_1, \ldots, x_n]$ homogeneous we define its *tropicalization*

$$\mathrm{Trop}(I) := \{w \in \mathbb{R}^n : \mathrm{in}_w(I) \not\ni \text{monomials}\}$$

$\mathrm{Trop}(I)$ has a fan structure:

$$v, w \in C^\circ \quad \Leftrightarrow \quad \mathrm{in}_v(I) = \mathrm{in}_w(I).$$

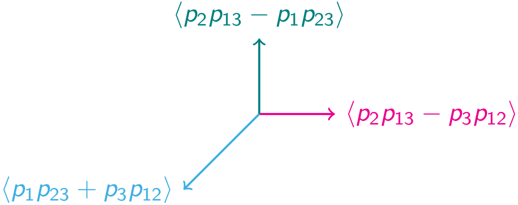
Notation: $\mathrm{in}_C(I) := \mathrm{in}_w(I)$ for $w \in C^\circ$.

Aim: For a projective variety X find an ideal I with $X = V(I)$ such that $\mathrm{Trop}(I)$ contains a maximal cone with associated ideal $\mathrm{in}_C(I)$ *binomial and prime*.

Example: $\mathcal{F}\ell_3$

We have $\mathcal{F}\ell_3 = V(I_3)$ with $I_3 = \langle p_2 p_{13} - p_1 p_{23} - p_3 p_{12} \rangle$.

Then $\text{Trop}(I_3) \subset \mathbb{R}^6/\mathbb{R}^2$ is 3-dimensional fan with 2-dimensional linear subspace $\mathcal{L}$ and three cones



$\rightsquigarrow$ all are *binomial and prime*!

Well-poised

Best case: $V(I)$ is *well-poised*, i.e. all initial ideals of maximal cones in $\mathrm{Trop}(I)$ are prime.

Example

This is true for

- $\mathcal{F}\ell_3 = V(I_3)$,
- $\mathrm{Gr}_2(\mathbb{C}^n) = V(I_{2,n})$ by Speyer–Sturmfels,
- for rational complexity-one T-varieties by Ilten–Manon.

$\rightsquigarrow$ In general well-poised is a lot to ask for.

Total positivity

Definition

An ideal $J \subset \mathbb{R}[x_1, \dots, x_n]$ *totally positive* if it does not contain any non-zero element of $\mathbb{R}_{\geq 0}[x_1, \dots, x_n]$.

Example

- $\langle p_1 p_{23} + p_3 p_{12} \rangle$ is <u>not</u> totally positive;
- $\langle p_2 p_{13} - p_3 p_{12} \rangle$ and $\langle p_2 p_{13} - p_3 p_{12} \rangle$ are totally positive.

Positively well-poised

We denote by $\text{Trop}^+(I) \subset \text{Trop}(I)$ the subfan with totally positive initial ideals [Speyer–Williams].

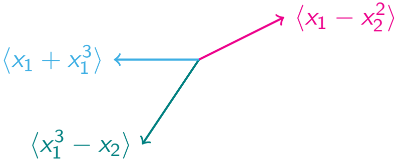
Definition

$V(I)$ is *positively well-poised* if all initial ideals of maximal cones in $\text{Trop}^+(I)$ are prime.

Example

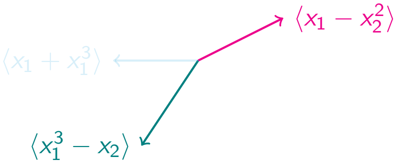
Take $I = \langle x_1 + x_1^3 - x_2^2 \rangle \subset \mathbb{C}[x_1, x_2]$.

Trop(I):



$\rightsquigarrow$ $V(I)$ is *not well-posed* as $\langle x_1 + x_1^3 \rangle$ is not prime.

Trop$^+$(I):



$\rightsquigarrow$ $V(I)$ is *positively well-poised*!

Example: Tropicalizing $\mathcal{F}\ell_4$

Theorem (B.–Lamboglia–Mincheva–Mohammadi)

The tropical flag variety $Trop(I_4)$ is a 6-dimensional fan in $\mathbb{R}^{14}/\mathbb{R}^3$ with 78 maximal cone:

- *72 maximal cones have binomial and prime initial ideals,*
- *6 maximal cones have binomial but not prime initial ideals.*

$\rightsquigarrow$ $V(I_4)$ is *not well-poised*.

$\text{Trop}^+(I_4) \subset \text{Trop}(I_4)$ consists of 14 maximal cones:

- 12 maximal cones have *binomial and prime* initial ideals,
- 2 maximal cones have *binomial but not prime* initial ideals.

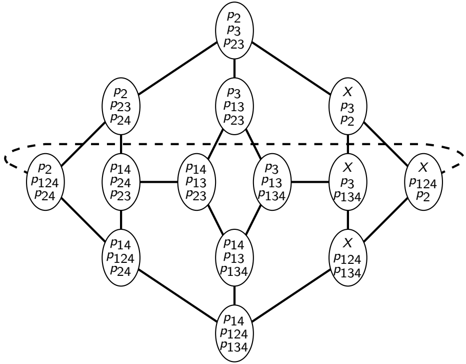
$\rightsquigarrow$ $V(I_4)$ is also not positively well-poised.

Example: Cluster structure $\mathcal{F}\ell_4$

The cluster algebra A_4 has 14 seeds, all of which contain the *frozen variables*

$$p_1, p_4, p_{12}, p_{34}, p_{123}, p_{234}$$

and additional 3 cluster variables:



Example: $\mathcal{F}\ell_4$

Observation: combinatorially can identify seeds of A_4 with maximal cones in $\mathrm{Trop}^+(I_4)$[3], but the toric degenerations do not match.

All cluster degenerations are toric while 2 degenerations of $\mathrm{Trop}^+(I_4)$ fail to be prime.

$\rightsquigarrow$ replace the Plücker ideal of $\mathcal{F}\ell_4$ by its *cluster ideal*: there exists an ideal $J_4 \subset \mathbb{C}[x, p_1, \ldots, p_{234}]$ with

$$A_4 \cong \mathbb{C}[x, p_1, \ldots, p_{234}]/J_4.$$

[3] Compare to Speyer–Williams tropical totally positive $\mathrm{Gr}(2, n)$.

Example: $\mathcal{F}\ell_4$

Trop(J_4) is 6-dimensional fan in $\mathbb{R}^{15}/\mathbb{R}^3$ with 105 maximal cones:

- 99 maximal cones have *binomial and prime* initial ideals,
- 6 maximal cones have *binomial but not prime* initial ideals.

$\rightsquigarrow$ $V(J_4)$ is still *not well-poised*.

Trop$^+$(J_4) $\subset$ Trop(J_4) has 14 maximal cones and

- 14 maximal cones have *binomial and prime* initial ideals.

$\rightsquigarrow$ $V(J_4)$ is *positively well-poised*!

Can identify maximal cones in Trop$^+$(J_4) with seeds in A_4 such that the associated toric degenerations coincide!

Example: $\mathcal{F}\ell_4$

The ideals are minimally generated as follows

$$
\begin{array}{cc}
I_4 & J_4 \\
p_3p_{24} - p_4p_{23} - p_2p_{34}, & p_3p_{24} - p_4p_{23} - p_2p_{34}, \\
p_3p_{14} - p_4p_{13} - p_1p_{34}, & p_3p_{14} - p_4p_{13} - p_1p_{34}, \\
p_2p_{14} - p_4p_{12} - p_1p_{24}, & p_2p_{14} - p_4p_{12} - p_1p_{24}, \\
p_2p_{13} - p_3p_{12} - p_1p_{23}, & p_2p_{13} - p_3p_{12} - p_1p_{23}, \\
p_{24}p_{134} - p_{34}p_{124} - p_{14}p_{234}, & p_{24}p_{134} - p_{34}p_{124} - p_{14}p_{234}, \\
p_{23}p_{134} - p_{34}p_{123} - p_{13}p_{234}, & p_{23}p_{134} - p_{34}p_{123} - p_{13}p_{234}, \\
p_{23}p_{124} - p_{24}p_{123} - p_{12}p_{234}, & p_{23}p_{124} - p_{24}p_{123} - p_{12}p_{234}, \\
p_{13}p_{124} - p_{14}p_{123} - p_{12}p_{134}, & p_{13}p_{124} - p_{14}p_{123} - p_{12}p_{134}, \\
p_{13}p_{24} - p_{14}p_{23} - p_{12}p_{34}, & p_{13}p_{24} - p_{14}p_{23} - p_{12}p_{34}, \\
p_4p_{123} - p_3p_{124} + p_2p_{134} - p_1p_{234}, & p_3p_{124} - x - p_4p_{123}, \\
 & p_2p_{134} - x - p_1p_{234}
\end{array}
$$

Both are prime ideals and $V(I_4) \cong V(J_4) \cong \mathcal{F}\ell_4$.

Thank you!

References

B Lara Bossinger. Computational data on the totally positive tropical flag variety. *available upon request*.

BFZ05 Arkady Berenstein, Sergey Fomin, and Andrei Zelevinsky. Cluster algebras. III. Upper bounds and double Bruhat cells. *Duke Math. J.* 126, no. 1, 1–52, 2005.

BLMM17 Lara Bossinger, Sara Lamboglia, Kalina Mincheva, and Fatemeh Mohammadi. Computing toric degenerations of flag varieties. In *Combinatorial algebraic geometry*, volume 80 of *Fields Inst. Commun.*, pages 247–281. Fields Inst. Res. Math. Sci., Toronto, ON, 2017.

FZ02 Sergey Fomin and Andrei Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529, 2002.

GHKK18 Mark Gross, Paul Hacking, Sean Keel, and Maxim Kontsevich. Canonical bases for cluster algebras. *J. Amer. Math. Soc.*, 31(2):497–608, 2018.

IM19 Nathan Ilten and Christopher Manon. Rational complexity-one T-varieties are well-poised. *Int. Math. Res. Not. IMRN* no. 13, 4198–4232, 2019.

MS15 Diane Maclagan and Bernd Sturmfels. *Introduction to tropical geometry*, volume 161 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2015.

Sco06 Joshua S. Scott. Grassmannians and cluster algebras. *Proc. London Math. Soc.* (3) 92, no. 2, 345–380, 2006.

SS04 David Speyer and Bernd Sturmfels. The tropical Grassmannian. *Advances in Geometry*, 4(3):389–411, 2004.

SW05 David Speyer and Lauren Williams. The tropical totally positive Grassmannian. *J. Algebraic Combin.* 22, no. 2, 189–210, 2005.