Cluster algebras for scattering amplitudes

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Motivation: scattering amplitudes

In quantum field theory physicist compute probabilities of particle interactions via scattering amplitudes.

The *scattering amplitud* is a function on the kinematic space that models the particle configuration, typically a generalized polylogarithm expressed as an iterated integral.

I want to tell you:

1. What kind of functions?
2. Defined on which spaces?
3. Where are the cluster algebras?

Goncharov polylogarithms [C. Duhr, H. Gangl, J.Rhodes JHEP10(2012)075]

For $n \geq 0$ and $a_i \in \mathbb{C}$ we define the *Goncharov polylogarithm* as

$$G(a_1, \ldots, a_n; z) := \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t), \quad G(; z) = 1 \; (G(0) = 0)$$

with $G(k\vec{a}; kx) = G(\vec{a}; x)$ for $a_n \neq 0, k \in \mathbb{C}^*$

Shuffle algebra: for $\Sigma(n_1, n_2) \subset S_{n_1+n_2}$ denoting shuffles

$$G(a_1, \ldots, a_{n_1}; x)\, G(a_{n_1+1}, \ldots, a_{n_1+n_2}; x) = \sum_{\sigma \in \Sigma(n_1, n_2)} G(a_{\sigma(1)}, \ldots, a_{\sigma(n_1+n_2)}; x),$$

Examples:

$$G(\vec{0}_n; x) = \frac{1}{n!} \log^n x, \qquad G(\vec{a}_n; x) = \frac{1}{n!} \log^n \left(1 - \frac{x}{a}\right),$$

$$G(\vec{0}_{n-1}, a; x) = -\mathrm{Li}_n\left(\frac{x}{a}\right), \qquad G(\vec{0}_n, \vec{a}_p; x) = (-1)^p\, S_{n,p}\left(\frac{x}{a}\right),$$

where $S_{n,p}$ denotes the Nielson polylogarithm.

Symbol calculus [C. Duhr, H. Gangl, J.Rhodes JHEP10(2012)075]

The differential structure

$$dG(a_{n-1},\ldots,a_1;a_n)=\sum_{i=1}^{n-1}G(a_{n-1},\ldots,\hat{a}_i,\ldots,a_1;z)d\log\frac{a_i-a_{i+1}}{a_i-a_{i-1}}$$

motivates the definition of the *symbol map*

$$\mathcal{S}(G(a_{n-1},\ldots,a_1;a_n))=\sum_{i=1}^{n-1}\mathcal{S}(G(a_{n-1},\ldots,\hat{a}_i,\ldots,a_1;z))\otimes\frac{a_i-a_{i+1}}{a_i-a_{i-1}}$$

applied iteratively it associates a tensor product (called *word*) in variables (called *letters*).

Distributivity:

$$C\otimes(a\cdot b)\otimes D=C\otimes a\otimes D+C\otimes b\otimes D$$

$$C\otimes a^n\otimes D=n\big(C\otimes a\otimes D\big)\,,\qquad n\in\mathbb{Z},$$

Shuffle product:

$$\mathcal{S}\big(G(a_1,\ldots,a_r;x)G(b_1,\ldots,b_s;y)\big)=\mathcal{S}\big(G(a_1,\ldots,a_r;x)\big)\,\text{Ш}\,\mathcal{S}\big(G(b_1,\ldots,b_s;y)\big)$$

Multiple polylogarithms [C. Duhr, H. Gangl, J.Rhodes JHEP10(2012)075]

Converging multiple polylogarithms have an expression as:

$$\mathrm{Li}_{m_1,\ldots,m_k}(x_1,\ldots,x_k) = \sum_{n_1<n_2<\cdots<n_k} \frac{x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k}}{n_1^{m_1} n_2^{m_2} \cdots n_k^{m_k}} = \sum_{n_k=1}^{\infty} \frac{x_k^{n_k}}{n_k^{m_k}} \sum_{n_{k-1}=1}^{n_k-1} \cdots \sum_{n_1=1}^{n_2-1} \frac{x_1^{n_1}}{n_1^{m_1}}.$$

They are related to Goncharov polylogarithms via

$$\mathrm{Li}_{m_1,\ldots,m_k}(x_1,\ldots,x_k) = (-1)^k \, G_{m_k,\ldots,m_1}\left(\frac{1}{x_k},\ldots,\frac{1}{x_1\ldots x_k}\right)$$

Example 0:

$$\mathrm{Li}_k(z) = \int_0^z \mathrm{Li}_{k-1}(t)\, d\log t, \quad \mathrm{Li}_1(z) = -\log(1-z) \tag{15}$$

clearly have

$$\mathrm{symbol}(\mathrm{Li}_k(z)) = -(1-z) \otimes \underbrace{z \otimes \cdots \otimes z}_{k-1 \text{ times}}. \tag{16}$$

Example 1: $G(a, b; x)$ [C. Duhr, H. Gangl, J.Rhodes JHEP10(2012)075]

Example 1: For a, b distinct and non zero one verifies

$$G(a,b;x) = \mathrm{Li}_2\left(\frac{b-x}{b-a}\right) - \mathrm{Li}_2\left(\frac{b}{b-a}\right) + \log\left(1-\frac{x}{b}\right)\log\left(\frac{x-a}{b-a}\right)$$

Combined with **Example 0** we may deduce:

$$\mathcal{S}\big(G(a,b;x)\big) = \left(1-\frac{x}{b}\right)\otimes\left(1-\frac{x}{a}\right) \; - \; \left(1-\frac{x}{b}\right)\otimes\left(1-\frac{b}{a}\right) + \left(1-\frac{x}{a}\right)\otimes\left(1-\frac{a}{b}\right)$$

Example 2: Scattering amplitude

[A. Goncharov, M. Spradlin, C. Vergu, A. Volovich. Phys. Rev. Lett. 105 (2010) 151605]

After manipulating the ($\mathcal{N} = 4$ super Yang-Mills six particle two loop) *scattering amplitude* we are left with the remainder function:

$$R_6^{(2)} = \sum_{i=1}^{3} \left(L_i - \frac{1}{2} \operatorname{Li}_4(-v_i) \right) - \frac{1}{8} \left(\sum_{i=1}^{3} \operatorname{Li}_2(-v_i) \right)^2 + \frac{1}{24} J^4 + \frac{\pi^2}{12} J^2 + \frac{\pi^4}{72},$$

in terms of the functions

$$L_i = \frac{1}{384} P_i^4 + \sum_{m=0}^{3} \frac{(-1)^m}{(2m)!!} P_i^m (\ell_{4-m}(x_i^+) + \ell_{4-m}(x_i^-)),$$

$$P_i = 2 \operatorname{Li}_1(-v_i) - \sum_{j=1}^{3} \operatorname{Li}_1(-v_j),$$

and

$$J = \sum_{i=1}^{3} \ell_1(x_i^+) - \ell_1(x_i^-),$$

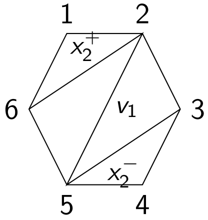
$$\ell_n(x) = \frac{1}{2} (\operatorname{Li}_n(-x) - (-1)^n \operatorname{Li}_n(-1/x)).$$

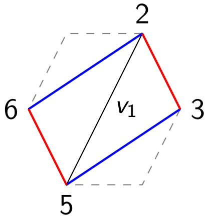
Example 2: symbol of a scattering amplitude

[A. Goncharov, M. Spradlin, C. Vergu, A. Volovich. Phys. Rev. Lett. 105 (2010) 151605]

$$
\begin{aligned}
v_1 &= \frac{\langle 35\rangle\langle 26\rangle}{\langle 23\rangle\langle 56\rangle}, & v_2 &= \frac{\langle 13\rangle\langle 46\rangle}{\langle 16\rangle\langle 34\rangle}, & v_3 &= \frac{\langle 15\rangle\langle 24\rangle}{\langle 45\rangle\langle 12\rangle}, \\
x_1^+ &= \frac{\langle 14\rangle\langle 23\rangle}{\langle 12\rangle\langle 34\rangle}, & x_2^+ &= \frac{\langle 25\rangle\langle 16\rangle}{\langle 56\rangle\langle 12\rangle}, & x_3^+ &= \frac{\langle 36\rangle\langle 45\rangle}{\langle 34\rangle\langle 56\rangle}, \\
x_1^- &= \frac{\langle 14\rangle\langle 56\rangle}{\langle 45\rangle\langle 16\rangle}, & x_2^- &= \frac{\langle 25\rangle\langle 34\rangle}{\langle 23\rangle\langle 45\rangle}, & x_3^- &= \frac{\langle 36\rangle\langle 12\rangle}{\langle 16\rangle\langle 23\rangle}
\end{aligned}
$$

Observe that combinatorially the symbol alphabet corresponds to *quadrilaterals in a hexagon:*

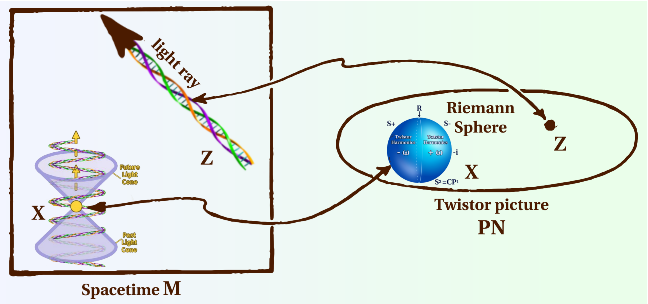




$$v_1 = \frac{\langle 35\rangle\langle 26\rangle}{\langle 23\rangle\langle 56\rangle}$$

Twistor theory

[R. Penrose. *Twistor algebra*. J. Math. Phys. 8, 345–366 (1967)]



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Momentum twistors

[A. Hodges. *Eliminating spurious poles from gauge-theoretic amplitudes* JHEP, 2013(5), 135]

A particle configuration in $\mathcal{N} = 4$ super Yang Mills may be represented by momentum twistors $Z_1, \ldots, Z_n \in \mathbb{C}P^3$. The system is parametrized by

$$\langle ijkl \rangle := \det(Z_i Z_j Z_k Z_l),$$

for $1 \leq i < j < k < l \leq n$ satisfying determinantal identities (Plücker relations). This configuration space is closely related to the Grassmannian $\mathrm{Gr}_{4,n}$ (up to scaling.

Example 2: For $n = 6$ we have $\mathrm{Gr}_{4,6} \cong \mathrm{Gr}_{2,6}$ and the symbol alphabet of $R_6^{(2)}$ consits of the $\mathcal{X}$ cluster variables (resp. $\hat{y}$-variables).

Symbol alphabet and cluster algebras

In $\mathcal{N} = 4$ super Yang–Mills the symbol alphabet of the scattering amplitude for n particles

$n \leq 7$ $\mathrm{Gr}_{4,n}$ has finitely many cluster variables and they determine the complete symbol alphabet

[Golden, Goncharov, Spradlin, Vergu, Volovich. JHEP 01 (2014) 091]

$n = 8$ $\mathrm{Gr}_{4,8}$ has infinitely many cluster variables, however the complete alphabet of 272 letters is determined by a finite collection of cluster variables and expressions obtained as limits along infinite mutation sequences of Kronecker type.

[Drummond, Foster, Gurdogan, Kalousios. JHEP 11 (2021) 071]

Criticism: $\mathcal{N} = 4$ SYM is a toy model *too far* from the real world modeled by the standard model

Question: Are cluster algebras relevant in other models?

Beyond $\mathcal{N} = 4$ super Yang–Mills

Introduce an *infinity twistor*, a line spanned by $Z_{n+1}, Z_{n+2} \in \mathbb{C}P^3$, breaking cyclic symmetry in the model and parametrise by

$$\langle ijkl \rangle := \det(Z_i Z_j Z_k Z_l), \quad \langle ij \rangle := \det(Z_i Z_j Z_{n+1} Z_{n+2}),$$

for $1 \leq i < j < k < l \leq n$ resp. $1 \leq i < j \leq n$.

This motivates us to define the *momentum twistor variety* as the subvariety of $\mathbb{P}^{\binom{n}{2}-1} \times \mathbb{P}^{\binom{n}{4}-1}$ given by the vanishing of determinantal identidies amoing the $\langle ijkl \rangle$ and $\langle ij \rangle$.

The momentum twistor variety $\mathcal{MT}_n$ is the *partial flag variety $\mathcal{F}_{2,4;n}$*.

Beyond $\mathcal{N} = 4$ super Yang–Mills

Let $\{p_1, \dots, p_n\} \subset \mathbb{R}^4$ be a configuration of n lightlike particles in QCD (standard model)

1. its helicity spinors are n pairs $(\lambda_i, \tilde{\lambda}_i) \in \mathbb{C}^{2\times 2}$ with $\lambda_i \tilde{\lambda}_i^T = 0$
2. define $\Lambda = [\lambda_1, \dots, \lambda_n], \tilde{\Lambda} = [\tilde{\lambda}_1, \dots, \tilde{\lambda}_n] \in \mathrm{Gr}_{2;n}$ with $\Lambda\tilde{\Lambda}^T = 0$

The *spinor helicity variety*

$$\mathcal{SH}_n := \{(\Lambda, \tilde{\Lambda}) \in \mathrm{Gr}_{2;n} \times \mathrm{Gr}_{2;n} : \Lambda\tilde{\Lambda}^T = 0\}.$$

is parametrized by 2×2 minors with $1 \leq i < j \leq n$

$$\langle ij \rangle := \det(\lambda_i \lambda_j) \quad \text{y} \quad [ij] := \det(\tilde{\lambda}_i \tilde{\lambda}_j)$$

satisfying *Plücker relations* and *momentum conservation*

$$\begin{aligned} 0 &= \langle ij\rangle\langle kl\rangle - \langle ik\rangle\langle jl\rangle + \langle il\rangle\langle jk\rangle = [ij][kl] - [ik][jl] + [il][jk] \\ 0 &= \sum_{s=1}^{n} \langle is\rangle[sj] \quad (\Leftrightarrow \Lambda\tilde{\Lambda}^T = 0) \end{aligned}$$

Spinor helicity variety

[Y.El Mazzouz, A.Pfister, and B.Sturmfels. 2406.17331]

Consider the map

$$\langle ij\rangle \mapsto P_{ij}, \quad \text{and} \quad [ij] \mapsto (-1)^{i+j-1} P_{[n]-ij}$$

where $[n]-ij := \{1,\ldots,n\}-\{i,j\}$. Momentum conservation also turns into Plücker relations

$$\begin{aligned} P_{ij}P_{kl} - P_{ik}P_{jl} + P_{il}P_{jk} &= 0 \\ P_{[n]-ij}P_{[n]-kl} - P_{[n]-ik}P_{[n]-jl} + P_{[n]-il}P_{[n]-jk} &= 0 \\ \sum_{s=1}^{n} (-1)^{s+j-1} P_{is}P_{[n]-js} &= 0 \end{aligned}$$

As $\dim \mathcal{SH}_n = 4(n-3) = \dim \mathcal{F}_{2,n-2;n}$ the map induces an isomorphism between the spinor helicity variety $\mathcal{SH}_n$ and the *partial flag variety* $\mathcal{F}_{2,n-2;n}$.

Cluster algebras

Theorem (C. Geiss, B. Leclerc, J. Schröer. Ann. Inst. Fourier 2008)

The homogeneous coordinate ring of a partial flag variety in its Plücker embedding $\mathbb{C}[\mathcal{F}_{d_1,\dots,d_k;n}]$ *is a cluster algebra.*

Both, the spinor helicity varieties by $\mathcal{F}_{2,n-2;n}$ and the momentum twistor varieties by $\mathcal{F}_{2,4;n}$ inherit these cluster structures.

Symbol alphabet for 5 particles in QCD

[L.B., J. Drummond, R. Glew. JHEP11(2023)002]

Analytic computations have found a symbol alphabet of 31 letters, one of which has unclear physical interpretation
[T. Gehrmann, J. M. Henn and N. A. Lo Presti. Phys. Rev. Lett. 116 (2016) 062001]

[D. Chicherin, J. Henn and V. Mitev. JHEP 05 (2018) 164]

Reformulated in spinor helicity variables up to cyclic symmetry it is

$$W_1 = \langle 12\rangle[12], \qquad W_6 = \langle 34\rangle[34] + \langle 45\rangle[45],$$

$$W_{11} = \langle 34\rangle[34] + \langle 35\rangle[35], \qquad W_{16} = \langle 13\rangle[13],$$

$$W_{21} = \langle 13\rangle[13] + \langle 34\rangle[34], \qquad W_{26} = \frac{\langle 45\rangle[51]\langle 12\rangle[24]}{[45]\langle 51\rangle[12]\langle 24\rangle},$$

$$W_{31} = [45]\langle 51\rangle[12]\langle 24\rangle - \langle 45\rangle[51]\langle 12\rangle[24].$$

$$\begin{aligned} W_6 &= \langle 34\rangle[34] + \langle 45\rangle[45] = \langle 12\rangle[12] - \langle 35\rangle[35], \\ W_{11} &= \langle 34\rangle[34] + \langle 35\rangle[35] = \langle 12\rangle[12] - \langle 45\rangle[45], \\ W_{21} &= \langle 13\rangle[13] + \langle 34\rangle[34] = \langle 14\rangle[14] - \langle 25\rangle[25], \end{aligned}$$

Symbol alphabet for 5 particles in QCD

[L.B., J. Drummond, R. Glew. JHEP11(2023)002]

$\mathcal{F}_{2,3;5}$ is of finite cluster type D_4 and has 16 cluster variables and 6 frozen variables.

- letters $W_1, \ldots, W_5, W_{16}, \ldots, W_{20}, W_{26}, \ldots, W_{30}$ are multiplicative combinations of Plücker coordinates (cluster variables of degree one)
- letters $W_6, \ldots, W_{10}, W_{11}, \ldots, W_{15}, W_{21}, \ldots, W_{25}$ are S_5 permutation copies of cluster variables of degree two
- spurious letter W_{31} is not recovered by the cluster algebra

Embeddings of cluster algebras

Let Q be a quiver and let Q' be obtained from Q by freezing and deleting vertices. Then Q' is called a *restricted quiver of Q*.

$\rightsquigarrow$ get induced notion of *restricted seeds* determining *cluster subpatterns* and *cluster subalgebras*.

Let p_J with $J \subset [n]$ with $|J| \in \{d_1, \ldots, d_k\}$ be Plücker coordinates of $\mathbb{C}[\mathcal{F}_{d_1,\ldots,d_k;n}]$, and P_J with $|J| = d_k$ be the Plücker coordinates of $\mathbb{C}[\mathrm{Gr}_{d_k,N}]$, $N = n + d_k - d_1$. Define for $|J| = d_j$:

$$p_J \mapsto P_{J \cup [n+1,\ldots,n+d_k-d_j]}$$

Theorem (L.B., Jianrong Li. arxiv:2408.14956)

The above map extends to an embedding of cluster algebras

$$\mathbb{C}[\mathcal{F}_{d_1,\ldots,d_k;n}] \hookrightarrow \mathbb{C}[\mathrm{Gr}_{d_k;N}]$$

Symbol alphabet for 6 particles in QCD

[A. Pokraka, M. Spradlin, A. Volovicha, H.-C. Weng, arxiv:2506.11895]
L.B., J. Drummond, E. Glew, Ö. Gürdoğan, and R. Wright. arxiv:2507.01015

Analytic computations have found a symbol alphabet of *289 letters* that split into *38 permutation classes*

[J. Henn, A. Matijašić, J. Miczajka, T. Peraro, et al. Phys.Rev.Lett. 135 (2025) 3, 031601]

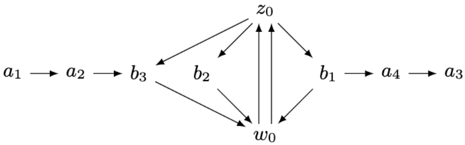
$\mathcal{F}_{2,4;6}$ is of affine type $D_6^{(1)}$, so it has infinitely many cluster variables

- we identify 54 cluster variables that are letters and give 11 permutation classes
- 32 letters are obtained as limits of infitine mutation sequences along a Kronecker subquiver from 16 different seeds
- Using the embedding $\mathbb{C}[\mathcal{F}_{2,4;6}] \hookrightarrow \mathbb{C}[\mathrm{Gr}_{4,8}]$ and known results for $\mathrm{Gr}_{4,8}$ we recover a total of 32 out of the 38 permutation classes

Infinite mutation sequences

[L.B., J. Drummond, E. Glew, Ö. Gürdoğan, and R. Wright. arxiv:2507.01015]

A seed for $\mathrm{Gr}_{4,8}$ (without frozens) that restricts to a seed for $\mathcal{F}_{2,4;6}$:



$$z_0 = p_{1236}\, p_{1578} - p_{1235}\, p_{1678} \qquad w_0 = p_{1356}$$
$$b_1 = p_{1236}\, p_{3578} - p_{1235}\, p_{3678} \qquad a_1 = p_{1345}$$
$$b_2 = p_{1256} \qquad a_2 = p_{1346}$$
$$b_3 = p_{1346}\, p_{1578} - p_{1345}\, p_{1678} \qquad a_3 = p_{1237}$$
$$a_4 = p_{1236}\, p_{4578} - p_{1235}\, p_{4678} + p_{1234}\, p_{5678}$$

Mutate w_0 and z_0, get a sequence z_i : $z_{i+2} z_i = b_1 b_2 b_3 F + z_{i+1}^2$.
Solve for z_i, get an expression containing the letter $\sqrt{\Delta}$ with

$$\Delta = p_{12}^2 p_{3456}^2 - 2p_{1256}\, p_{12} p_{3456}\, p_{34} + p_{1256}^2 p_{34}^2$$
$$- 2p_{1234}\, p_{12} p_{3456}\, p_{56} - 2p_{1234}\, p_{1256} p_{3478}\, p_{56} + p_{1234}^2 p_{56}^2$$

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