Cluster varieties with coefficients

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Motivation

- Algebraic: Fomin-Zelevinsky x-pattern and y-pattern. x-pattern can be endowed with coefficients: what about y-pattern?
- Geometric: Gross-Hacking-Keel-Kontsevich introduce flat family $\mathcal{A}_{\text{prin}}$: what is the cluster dual $\mathcal{X}$-analogue?
- Toric degenerations: GHKK degenerate Grassmannians to toric varieties using $\mathcal{A}$-structure. Rietsch-Williams degenerate Grassmannians to toric varieties using $\mathcal{X}$-structure: how are they related?

Mutations

$N \cong \mathbb{Z}^n$ lattice, $\{\cdot,\cdot\} : N \times N \to \mathbb{Z}$, $M = N^*$

$$\begin{aligned} \mu_{(n,m)} : T_N &\dashrightarrow T_N \\ \mu^*_{(n,m)}(z^{m'}) &= z^{m'}(1+z^m)^{m'(n)}. \end{aligned}$$

Let $\{e_i\}$ basis of N, $\{f_i\}$ dual basis of M

$$T_N := \mathrm{Spec}(\mathbb{C}[M]) = \mathrm{Spec}\mathbb{C}[z^{\pm f_1}, \ldots, z^{\pm f_n}]$$

$\mathcal{A}$-cluster varieties

fix $s_0 = \{e_i\}$ basis of $N, v_i := \{e_i, \cdot\} \in M$
$\rightsquigarrow s = \{e_i^s\}$ new basis of N by certain pseudoreflections

Definition ($\mathcal{A}$-cluster mutation)

$$\begin{aligned} \mu_{(-e_k,v_k)} : T_{N,s_0} &\dashrightarrow T_{N,s} \\ \mu^*_{(-e_k,v_k)}(z^{m'}) &= z^{m'}(1+z^{v_k})^{m'(-e_k)} \end{aligned}$$

$$\mathcal{A} := \bigcup_{s \sim s_0} T_{N,s} \text{ glued by } \mathcal{A}\text{-mutations}$$

Example: $\mathcal{A}$ in case $\mathtt{A}_2$

z^{f_1}

z^{f_2}

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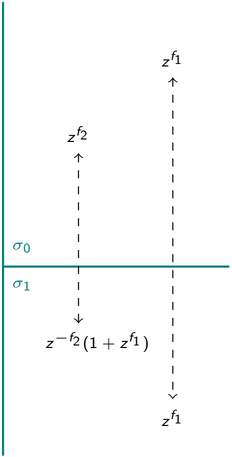

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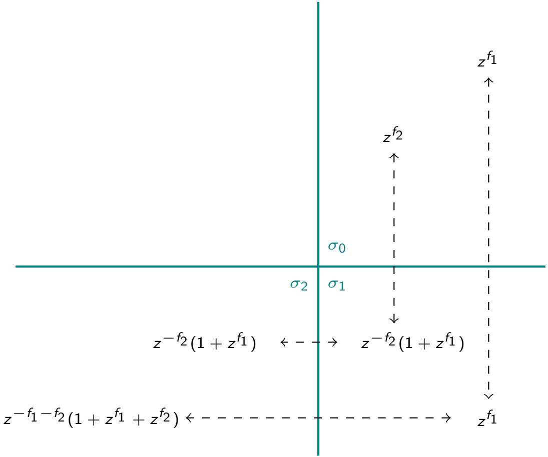

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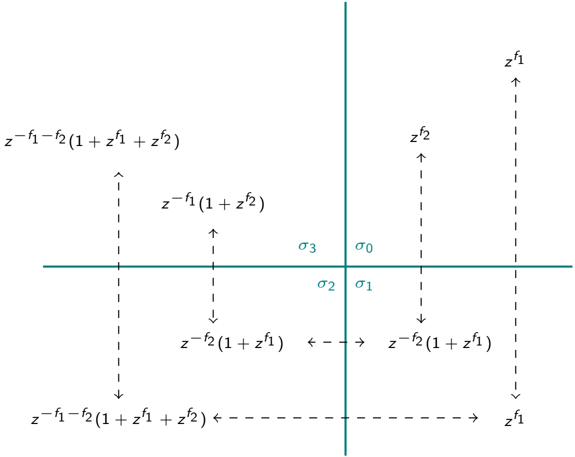

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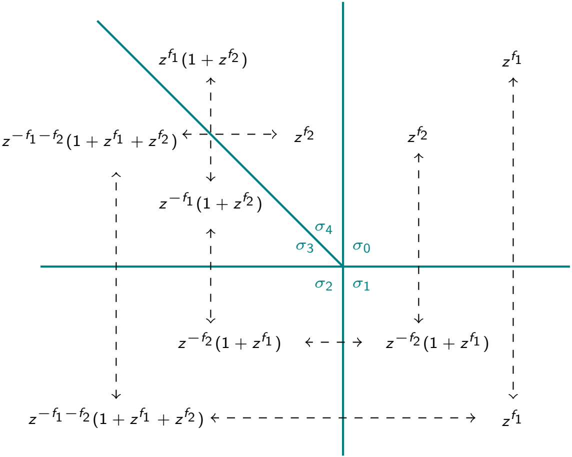

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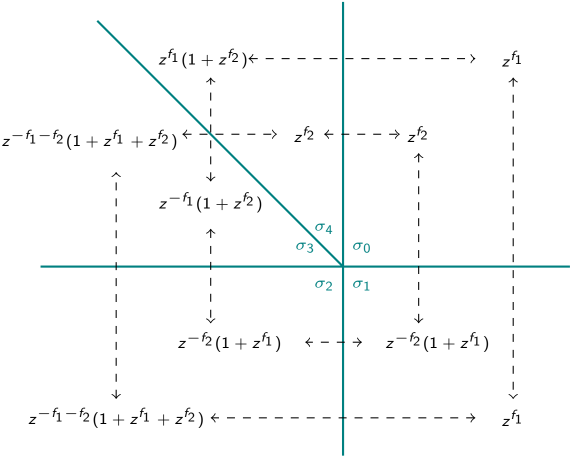

$\mathcal{X}$-cluster varieties

Exchange M and N

$$\begin{aligned} \mu_{(m,n)} : T_M &\dashrightarrow T_M \\ \mu^*_{(m,n)}(z^{n'}) &= z^{n'}(1+z^n)^{n'(m)}. \end{aligned}$$

Definition ($\mathcal{X}$-cluster mutation)

$s_0 = \{e_i\}$ basis of N, $v_i := \{e_i, \cdot\} \in M$

$$\begin{aligned} \mu_{(v_k,e_k)} : T_{M,s_0} &\dashrightarrow T_{M,s} \\ \mu^*_{(v_k,e_k)}(z^{n'}) &= z^{n'}(1+z^{e_k})^{n'(v_k)} \end{aligned}$$

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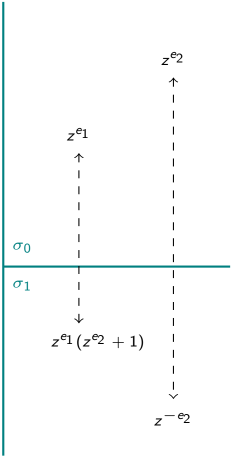

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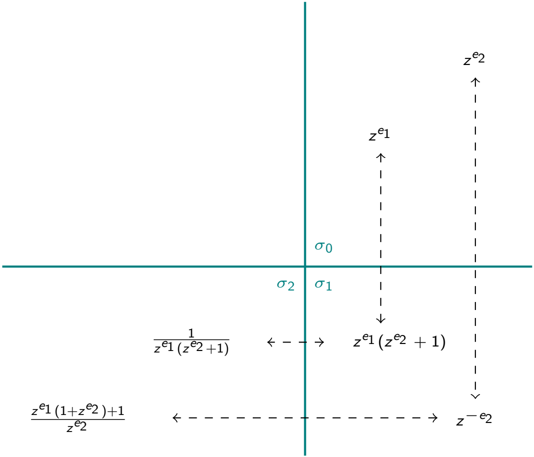

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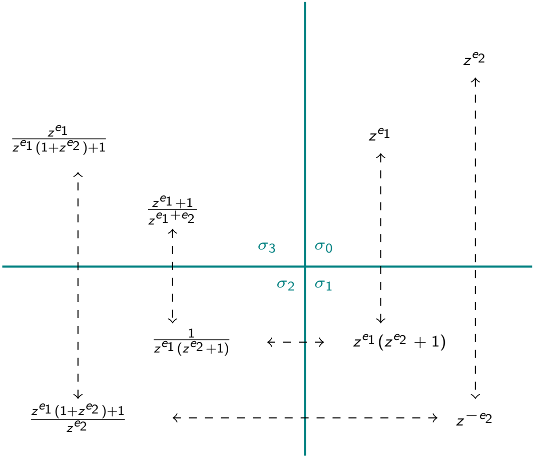

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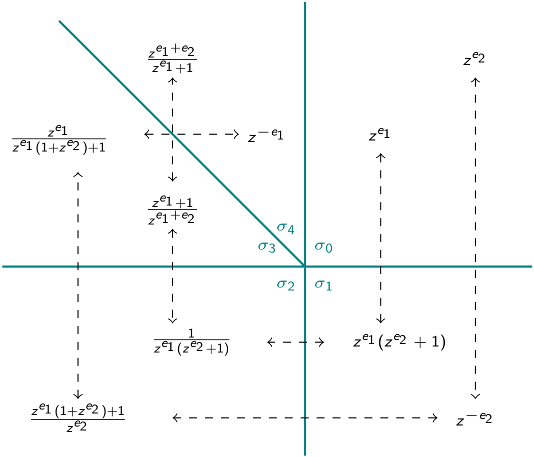

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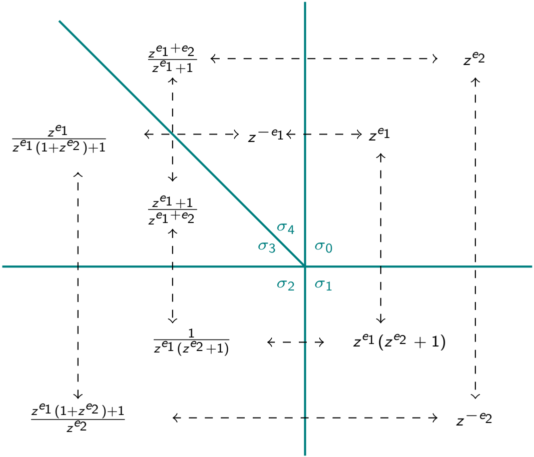

Cluster Varieties with coefficients

$R := \mathbb{C}[t_1, \ldots, t_n]$,
$c \in \mathbb{Z}^n$, $c = c_+ - c_-$

$$\begin{array}{rcl} \mu_{(n,m),c} : T_N(R) & \dashrightarrow & T_N(R) \\ \mu^*_{(n,m),c}(\tilde{z}^{m'}) & = & \tilde{z}^{m'}(t^{c_+} + t^{c_-}\tilde{z}^m)^{m'(n)} \end{array}$$

Definition (cluster mutation with coefficients)

$\mathcal{A}$-cluster mutation with coefficients:

$$\mu_{(-e_k, v_k), c_k} : T_{N,s_0}(R) \dashrightarrow T_{N,s}(R).$$

$\mathcal{X}$-cluster mutation with coefficients:

$$\mu_{(v_k, e_k), c_k} : T_{M,s_0}(R) \dashrightarrow T_{M,s}(R).$$

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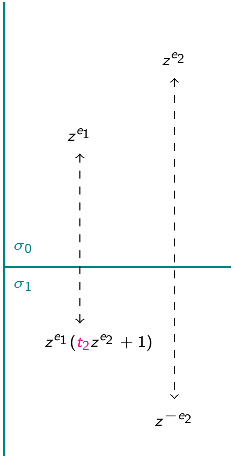

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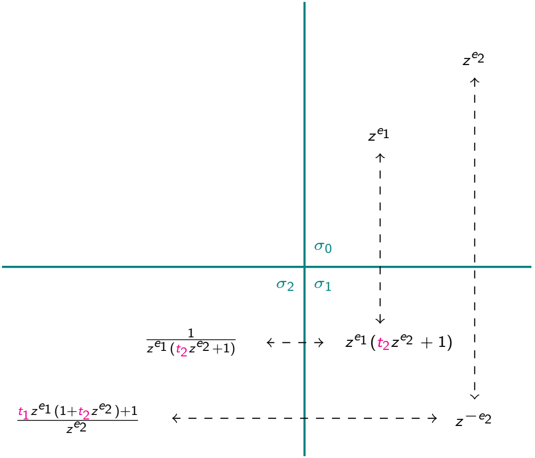

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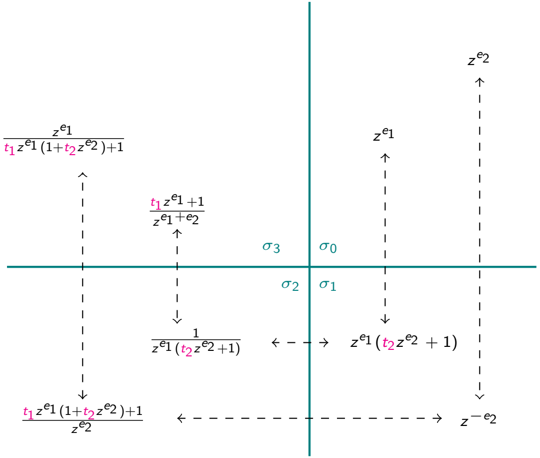

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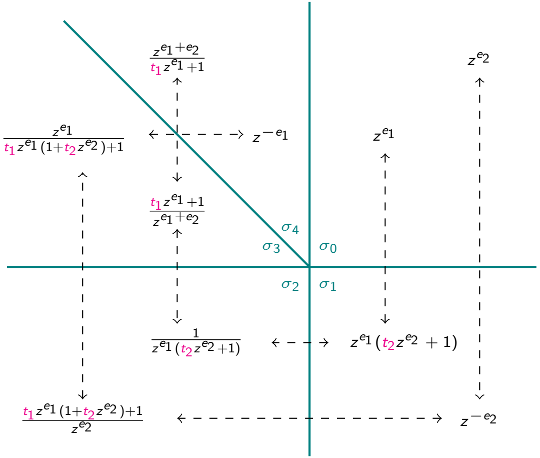

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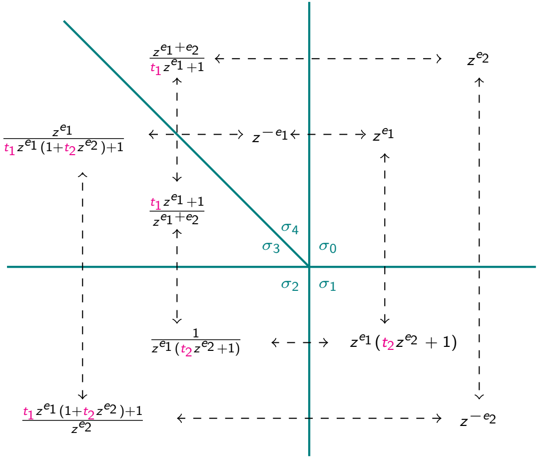

Flat families

Lemma

$T_N(R) = T_N \times_{\mathbb{C}} \mathbb{A}^n \to \mathbb{A}^n$ *extends to flat family*

$$\mathcal{A}_{prin} := \bigcup_{s \sim s_0} T_{N,s}(R) \to \mathbb{A}^n$$

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$$\mathfrak{X} := \bigcup_{s \sim s_0} T_{M,s}(R) \to \mathbb{A}^n$$

$$\begin{array}{ccccc} \mathcal{A}_{\mathrm{prin}} & & & & \mathfrak{X} \\ \pi_{\mathcal{A}} & \searrow & & \swarrow & \pi_{\mathcal{X}} \\ & & \mathbb{A}^n & & \end{array}$$

$\mathbf{g}$-vectors

Theorem (FZ - Laurent phenomenon)

$\tilde{z}^{f_i^s} \in R[\tilde{z}^{\pm f_1}, \ldots, \tilde{z}^{\pm f_n}]$ *is a global function on* $\mathcal{A}_{prin}$.

Theorem (GHKK)

- *In* $\pi_{\mathcal{A}}^{-1}(1) = \mathcal{A}$:
$$\tilde{z}^{f_i^s}|_{t=1} = z^{f_i^s} \in \mathbb{C}[\mathcal{A}],$$
a global function on $\mathcal{A}$.
- *In* $\pi_{\mathcal{A}}^{-1}(0) = T_N$:
$$\tilde{z}^{f_i^s}|_{t=0} = z^{\mathbf{g}_i^s} \in \mathbb{C}[M],$$
a character of T_N.
- *The* $\mathbf{g}_i^s \in M$ *form a simplicial fan called the* $\mathbf{g}$*-fan.*

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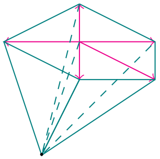

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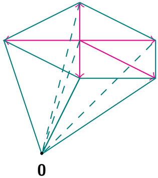
$$\begin{array}{c} \overline{\mathcal{A}}_{\mathrm{prin}} \\ \downarrow \pi_{\overline{\mathcal{A}}} \\ \mathbb{A}^n \end{array}$$



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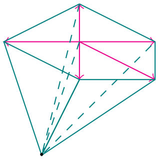
$$\pi_{\overline{\mathcal{A}}}^{-1}(1) = \overline{\mathcal{A}}$$



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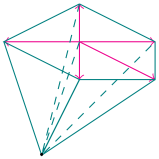
$$\pi_{\overline{\mathcal{A}}}^{-1}(1) = \overline{\mathcal{A}}$$

$$\pi_{\overline{\mathcal{A}}}^{-1}(0) = \mathrm{TV}(P)$$



$\mathcal{X}$-compactifications

$z^{e_i^s}$ local function on $\mathcal{X}$ $\rightsquigarrow$ can not allow $z^{e_i^s} = 0$
compactify locally: replace T_M by $\mathbb{A}_M = \mathbb{A}^n$

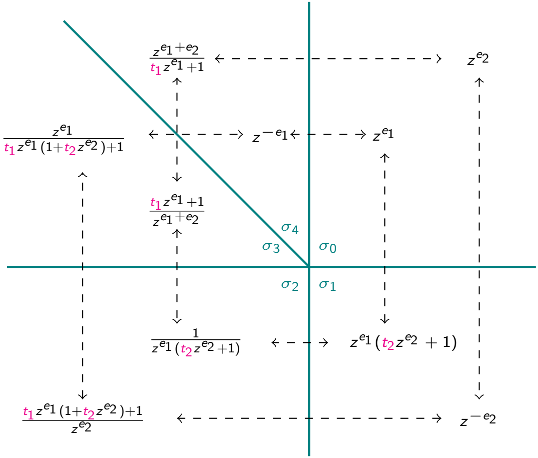
Theorem (BFMN)

$\mathcal{X}$-gluing with coefficients yields a flat family

$$\widehat{\mathfrak{X}} := \bigcup_{s \sim s_0} \mathbb{A}_{M,s}(R) \to \mathbb{A}^n$$

with $\pi^{-1}_{\widehat{\mathcal{X}}}(1) = \widehat{\mathcal{X}}$ Fock-Goncharov special completion, and $\pi^{-1}_{\widehat{\mathcal{X}}}(0) = TV(\mathbf{g}\text{-fan})$.

Example: $\widehat{\mathcal{X}}$ in case A_2



Thank you!

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