Toric degenerations: embeddings and projections

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Definition

Definition

Let X be a projective variety. A *toric degeneration* of X is a flat morphism $\xi : \mathfrak{X} \to \mathbb{A}^1$ with generic fibre isomorphic to X and special fibre $\xi^{-1}(0)$ a toric variety.

Examples:

1. an *abstract* degeneration, *e.g.* a toric scheme over $\mathbb{A}^1 = \mathrm{Spec}(k[t])$;
2. an *embedded* degeneration, *e.g.* $\mathfrak{X} = V(xy - x^2 + ty^2) \subset \mathbb{P}^1_{x:y} \times \mathbb{A}^1_t$;
3. a toric degeneration *admits a projection* if it is an embedded toric degeneration with a projection $\xi^{-1}(1) \twoheadrightarrow \xi^{-1}(0)$.

From abstract to embedded toric degenerations

Idea: If the toric fibre $\xi^{-1}(0)$ has a very ample line bundle, can we extend this embedding to all of the family?

Conjecture (Takuya Murata)

If a toric degenerations $\xi : \mathfrak{X} \to \mathbb{A}^1$ is proper and $\mathcal{L}$ is an invertible flat $\mathcal{O}_{\mathfrak{X}}$-module such that $\mathcal{L}|_{\xi^{-1}(0)}$ is very ample, then ξ it is an embedded degeneration; i.e. there exists an embedding $\mathfrak{X} \hookrightarrow \mathbb{P}^N \times \mathbb{A}^1$ such that

$$\begin{array}{ccc} \mathfrak{X} & \hookrightarrow & \mathbb{P}^N \times \mathbb{A}^1 \\ & {\scriptstyle\xi} \searrow & \downarrow \\ & & \mathbb{A}^1 \end{array}$$

is a commutative diagram.

Embedded toric degenerations

Given an embedded toric degeneration

$$\begin{array}{ccc} \mathfrak{X} & \hookrightarrow & \mathbb{P}^N \times \mathbb{A}^1 \\ & {\scriptstyle \xi} \searrow & \downarrow \\ & & \mathbb{A}^1 \end{array}$$

we have $\mathfrak{X} = \mathrm{Proj}(\mathfrak{R})$ for some Noetherian $k[t]$-algebra $\mathfrak{R}$. Then the generic fibre is

$$\xi^{-1}(1) = X = \mathrm{Proj}(R)$$

where $R := \mathfrak{R}/(t-1)\mathfrak{R}$. Similarly, the special fibre is $\xi^{-1}(0) = X_0 = \mathrm{Proj}(R_0)$ where $R_0 := \mathfrak{R}/t\mathfrak{R}$.

Assumption: X and X_0 are irreducible, so R is a positively graded domains and R_0 is a finitely generated algebra of a graded semigroup with identity.

[KM19]/[Mur20]: May assume $\mathfrak{R}$ is the *Rees algebra of a valuation* on R.

Toric degenerations from valuations

Let $R = \bigoplus_{i \geq 0} R_i$ be a graded k-algebra and domain. A *valuation* on R is a map $\nu : R \setminus \{0\} \to (\mathbb{Z}^d, <)$ such that for all $f, g \in R \setminus \{0\}$ and $c \in k$

$$\nu(fg) = \nu(f) + \nu(g), \quad \nu(cf) = \nu(f), \quad \nu(f+g) \geq \min{}_{<}\{\nu(f), \nu(g)\}$$

Notice: $S := \text{im}(\nu)$ is a semigroup.

Moreover, ν induces a filtration on R: for every $m \in \mathbb{Z}^d$

$$F_m := \{f \in R : \nu(f) \leq m\} \quad \text{and} \quad F_{<m} := \{f \in R : \nu(f) < m\}.$$

Proposition

If $F_m/F_{<m}$ is at most one-dimensional for all $m \in \mathbb{Z}^d$ (for example if $\text{rank}(S) = \dim(R)$, *i.e.* ν is *full-rank*) then

$$\text{gr}_\nu(R) \cong k[S].$$

A vector space basis $\mathbb{B}$ of R is *adapted to ν* if $\mathbb{B} \cap F_m$ is a vector space basis for all m.

Toric degenerations from valuations

Theorem (David Anderson)

Let $\nu : R \setminus \{0\} \to \mathbb{Z}^d$ be a full-rank valuation with finitely generated value semigroup S. Then there exists a toric degeneration of $X = Proj(R)$ with special fibre $X_0 = Proj(k[S])$ defined by the Rees algebra of ν*:*

$$\mathfrak{R} = \bigoplus_{i \geq 0} t^i F_{\leq i},$$

where $F_{\leq i} = \cup_{\pi(m) \leq i} F_m$ for a suitable projection $\pi : \mathbb{Z}^d \to \mathbb{Z}$.

$\mathfrak{R}$ is a flat $k[t]$-algebra with

$$\mathfrak{R}/(t-1)\mathfrak{R} = R \quad \text{and} \quad \mathfrak{R}/t\mathfrak{R} = \mathrm{gr}_\nu(R).$$

Equations for embedded toric degenerations

The polytope defining the (normalization of the) toric variety Proj($k[S]$) is the *Newton–Okounkov polytope*

$$\Delta(R,\nu) := \overline{\text{conv}\left(\bigcup_{i>0}\left\{\frac{\nu(f)}{i} : f \in R_i\right\}\right)} \subset \mathbb{R}^d.$$

Hence, we can compute equations for $X_{\Delta(R,\nu)} = \bar{X}_0$ from $\Delta(R,\nu)$.

(proper) abstract toric degeneration	$\rightsquigarrow$	(embedded) toric degeneration by a valuation	$\rightsquigarrow$	equations for the normalization of X_0

Question: How about equations for X and the family $\mathfrak{X}$?
$\rightsquigarrow$ can be obtained using *Gröbner theory*.

Gröbner degenerations

Let $k = \bar{k}$ with $\mathrm{char}(k) = 0$ and $R = k[x_1, \dots, x_n]/I$ for I homogeneous.

For every $w \in \mathbb{R}^n$ we have the *initial ideal* $\mathrm{in}_w(I) := (\mathrm{in}_w(f) : f \in I)$, for example $\mathrm{in}_{(1,1)}(xy - x^2 + y) = xy - x^2$, and a flat family

$$\xi_w : \mathfrak{X} \to \mathbb{A}^1$$

with generic fibre $\mathrm{Proj}(R)$ and special fibre $\mathrm{Proj}(R_w)$, where $R_w := k[x_1, \dots, x_n]/\mathrm{in}_w(I)$.

Definition

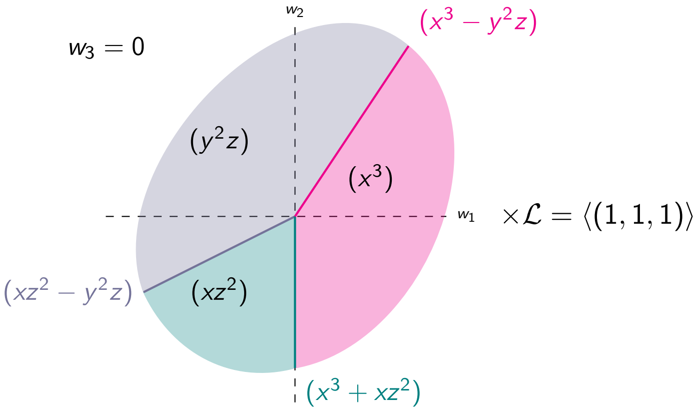
The *Gröbner fan* $\mathrm{GF}(I)$ of I is $\mathbb{R}^n$ with fan structure

$$v, w \in C^\circ \quad \Leftrightarrow \quad \mathrm{in}_v(I) = \mathrm{in}_w(I)$$

The *tropicalization* $\mathcal{T}(I)$ of I is the closed subfan of $\mathrm{GF}(I)$ consisting of those w for which $\mathrm{in}_w(I)$ contains no monomials.

Example

Take $I = (x^3 + xz^2 - y^2z) \subset \mathbb{C}[x, y, z]$. Then $GF(I)$ is $\mathbb{R}^3$ with the fan structure below and $\mathcal{T}(I)$ is its 1-skeleton.



Correspondence Theorem and Corollary

Theorem (L.B.'20, K.Kaveh–C.Manon '19)

Let R be a positively graded algebra and domain, $\nu : R \setminus \{0\} \to \mathbb{Z}^d$ full-rank valuation with finitely generated value semigroup. Then there exists an isomorphism of graded algebras

$$k[x_1, \dots, x_n]/I \cong R$$

such that Anderson's toric variety $Proj(k[S])$ is isomorphic to the toric variety of a Gröbner toric degeneration for some $w \in \mathcal{T}(I) \subset \mathbb{R}^n$:

$$Proj(k[S]) \cong Proj(R_w).$$

Corollary

The value semigroup $S \subset \mathbb{Z}^d$ is isomorphic to a semigroup $S' \subset \mathbb{Z}^d_{\geq 0}$.

Example: projected toric degenerations

Example

Consider the toric degeneration

$$\mathfrak{X} = V(y^2z - x^3 - txz^2) \subset \mathbb{P}^2_{x:y:z} \times \mathbb{A}^1_t$$

of the elliptic curve $X = V(y^2z - x^3 - xz^2)$ to the toric variety $X_0 = V(y^2z - x^3)$. The projection $X \to \mathbb{P}^1$ given by $[x : y : z] \mapsto [y : z]$ composed with the normalization map $\mathbb{P}^1 \to X_0$ defines a projection

$$X \twoheadrightarrow X_0.$$

Algebraically, this corresponds to an embedding of the semigroup algebra $R_0 = k[x, y, z]/(y^2z - x^3)$ into $R = k[x, y, z]/(y^2z - x^3 - xz^2)$.

Question: Which (embedded) toric degenerations admit such a projection?

Toric subalgebras

Let $\nu : R \setminus \{0\} \to \mathbb{Z}^d$ be a full-rank valuation with finitely generated semigroup S.

Algebraically, we are looking for an embedding of $k[S]$ as a *toric subalgebra* into R.

Idea: Map the basis S of $k[S]$ onto basis elements of R.

Example

In the above example, R has a k-basis $\mathbb{B} = \{x^a y^b z^c : a < 3\}$. The semigroup S defining R_0 is generated by $(1,0), (1,1), (1,3) \subset \mathbb{N} \times \mathbb{Z}$. So we may embed

$$k[S] \hookrightarrow R, \quad \chi^{(m,n)} \mapsto y^m z^n \in \mathbb{B}.$$

This map is neither graded nor finite, but it defines a dominant map

$$\mathrm{Spec}(R) \to \mathrm{Spec}(k[S])$$

$\rightsquigarrow$ these maps are not too hard to find, e.g. in cluster algebras.

Example: cluster algebras

A *cluster algebra* A is a commutative algebra generated recursively by

- *seeds*: maximal algebraically independent sets whose elements are called *cluster variables*, that are related to each other via
- *mutation*: an operation that creates a new seed from a given one by replacing one cluster variable by a binomial with positive coefficients in the other cluster variables.

The monomials in cluster variables of one seed are called *cluster monomials* and they are linearly independent in A.

Cluster algebras and valuations

Proposition (L.B.–M.Cheung–T.Magee–A.Nájera Chávez)

Let A be a cluster algebra that satisfies the *full Fock–Goncharov conjecture*. For every seed s there exists a full-rank valuation

$$g_s : A \setminus \{0\} \to \mathbb{Z}^d$$

with finitely generated semigroup. The associated Newton–Okounkov polytope $\Delta(A, g_s)$ is the tropicalization of *Gross–Hacking–Keel–Kontsevich's superpotential* for the associated cluster variety.

The Proposition applies to, for example, Grassmannians, flag varieties, configuration spaces, the del Pezzo surface of degree 5 ...

Cluster algebras and toric degenerations

Corollary (L.B.–Takuya Murata)

The toric degeneration of Spec(A) induced by $g_s : A \setminus \{0\} \to \mathbb{Z}^d$ admits a dominant map

$$Spec(A) \to \tilde{X}_{\Delta(A,g_s)}.$$

Idea of Proof: Consider the cluster variables $x_1, \dots, x_d$ of the seed s. They form a maximal algebraically independent set and all monomials in these variables are part of a k-basis $\mathbb{B}$ for A, $\rightsquigarrow$ *cluster monomials*.

Want to map the elements χ^m for $m \in \mathbb{Z}^d$ of the semigroup S to monomials $x_1^{m_1} \cdots x_d^{m_d}$. However, to do this we need $m \in \mathbb{Z}^d_{\geq 0}$. By the *Correspondence Corollary* S is isomorphic to a monoid $S' \subset \mathbb{Z}^d_{\geq 0}$. So we define

$$k[S] \cong k[S'] \hookrightarrow A, \quad \text{where} \quad \chi^m \mapsto \chi^{m'} \mapsto x_1^{m'_1} \cdots x_d^{m'_d} \in \mathbb{B}. \qquad \blacksquare$$

Standard monomial bases

More generally we may replace *cluster monomials* by *standard monomials*.

Definition

Let $<$ be a *monomial term order* on $k[x_1, \ldots, x_n]$, *i.e.* is a total order on the monomials in $k[x_1, \ldots, x_n]$ with $\mathrm{x}^a < \mathrm{x}^b$ implies $\mathrm{x}^{a+c} < \mathrm{x}^{b+c}$ for all $a, b, c \in \mathbb{Z}^n_{\geq 0}$. Then for an ideal $I \subset k[x_1, \ldots, x_n]$ its *initial ideal with respect to* $<$ is $\mathrm{in}_<(I) := (\mathrm{in}_<(f) : f \in F)$ where $in_<(f)$ is the $<$-maximal term in f.

It is not hard to see that $\mathrm{in}_<(I)$ is a monomial ideal. It defines a *standard monomial basis* for R

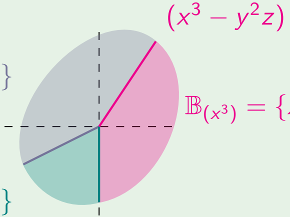
$$\mathbb{B}_< := \{\bar{\mathrm{x}}^m \in R : \mathrm{x}^m \notin \mathrm{in}_<(I)\}.$$

In fact, the maximal cones in the Gröbner fan $\mathrm{GF}(I)$ correspond to monomial initial ideals of form $\mathrm{in}_<(I)$.

Standard monomial bases for the elliptic curve

Example ($I = (x^3 + xz^2 - y^2z)$)

$\mathbb{B}_{(y^2z)} = \{x^a y^b z^c : b < 2 \text{ or } c < 1\}$



$\mathbb{B}_{(x^3)} = \{x^a y^b z^c : a < 3\}$

$\mathbb{B}_{(xz^2)} = \{x^a y^b z^c : a < 1 \text{ or } c < 2\}$

For $R_0 = k[x, y, z]/(x^3 - yz^2)$ we have two *adapted bases* $\mathbb{B}_{(x^3)}$ and $\mathbb{B}_{(yz^2)}$. For every choice of maximal algebraically independent set of generators in R_0 its monomials are standard in $\mathbb{B}_{(x^3)}$ *or* $\mathbb{B}_{(yz^2)}$:

$$\begin{aligned}
\{x, y\} &\rightsquigarrow x^a y^b \in \mathbb{B}_{(y^2z)} && \text{as } c < 1,\\
\{x, z\} &\rightsquigarrow x^a z^c \in \mathbb{B}_{(y^2z)} && \text{as } b < 2,\\
\{y, z\} &\rightsquigarrow y^b z^c \in \mathbb{B}_{(x^3)} && \text{as } a < 3.
\end{aligned}$$

From embedded toric degenerations to projections

Conjecture (L.B.–Takuya Murata)

Given an embedded toric degeneration $\mathfrak{X} \to \mathbb{A}^1$ determined by a full-rank valuation $\nu : R \setminus \{0\} \to \mathbb{Z}^d$ with finitely generated semigroup S there exists an embedding $k[S] \hookrightarrow R$ inducing a projection

$$Proj(R) \twoheadrightarrow Proj(k[S]).$$

Strategy of proof:

1. Use the *Correspondence Theorem*, so that $R = k[x_1, \dots, x_n]/I$ and there is a cone $\tau \in \mathcal{T}(I)$ with $k[S] = k[x_1, \dots, x_n]/\mathrm{in}_\tau(I)$;
2. fix a maximal algebraically independent set $s = \{x_{i_1}, \dots, x_{i_d}\}$ in $k[S]$;
3. refine τ by a term order such that monomials in s are standard;
4. adjust the grading of S and map S onto standard monomials;
5. check that a suitable localization $(k[S]_f)_0 \hookrightarrow R_f$ is finite.

Adjusting the grading of S

Example

Recall $S = \langle (1,0), (1,1), (1,3) \rangle_{\geq 0}$ is graded by $\deg(a,b) = a$, but in R monomials $y^a z^b$ have degree total $a + b$. Se we embed S into a semigroup S' graded by total degree:

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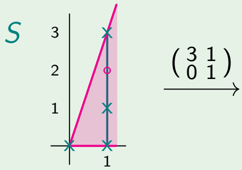

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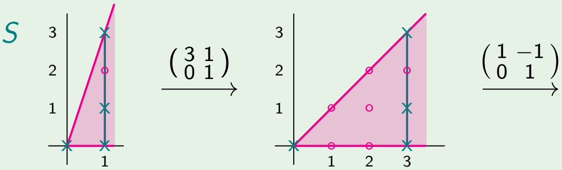

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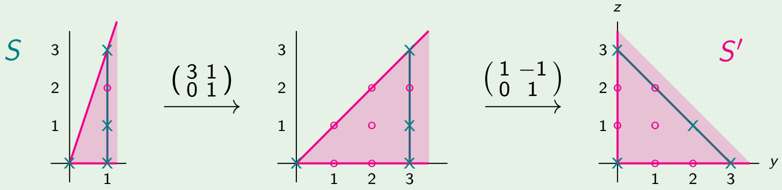

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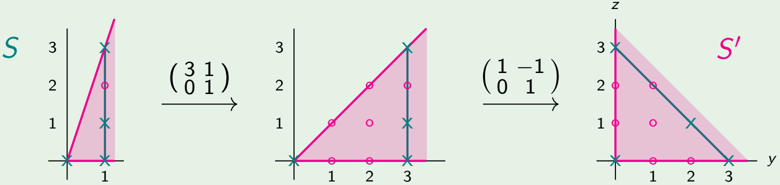

Recall that $y^a z^b \in R$ are standard monomials. So we may embed

$$k[S] \hookrightarrow R \quad \text{by} \quad (1,0) \mapsto y^3, \quad (1,1) \mapsto y^2 z \quad \text{and} \quad (1,3) \mapsto z^3.$$

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Proof of the Correspondence Theorem

We call a set of algebra generators $b_1, \ldots, b_n$ of R a *Khovanskii basis* for ν if $\nu(b_1), \ldots, \nu(b_n)$ generate S.

Idea of Proof: Choose a finite Khovanskii basis $b_1, \ldots, b_n \in R$. Take

$$\pi : k[x_1, \ldots, x_n] \to R, \quad x_i \mapsto b_i$$

and $I := \ker(\pi)$. Then by [B, Main Theorem] exits $w \in \mathcal{T}(I)$ such that

$$in_w(I) \quad \text{is toric} \quad \Leftrightarrow \quad S \quad \text{is finitely generated.}$$

Moreover, $k[S] \cong k[x_1, \ldots, x_n]/in_w(I)$. ■

Algorithm for w: Input: Khovanskii basis for ν; Output: w

1. compute $\nu(b_i)$ for all i;
2. for a suitable projection $\pi : \mathbb{Z}^d \to \mathbb{Z}$ compute
   $w = (\pi(\nu(b_1)), \ldots \pi(\nu(b_n)) \in \mathbb{R}^n$.