

Cluster algebras for scattering amplitudes

Lara Bossinger

arxiv:2507.01015, arxiv:2408.14956, JHEP11(2023)002

with James Drummond, Ross Glew, Ömer Gürdoğan, Jianrong Li and Rowan Wright

Universidad Nacional Autónoma de México, IM-Oaxaca



Ciencia y Tecnología
CONAHCyT

ARTA X Cologne, September 3 2025

Motivation: scattering amplitudes

In quantum field theory physicist compute probabilities of particle interactions via scattering amplitudes.

The *scattering amplitud*e is a function on the kinematic space that models the particle configuration, typically a generalized polylogarithm expressed as an iterated integral.

I want to tell you:

- 1 What kind of functions?
- 2 Defined on which spaces?
- 3 Where are the cluster algebras?

Goncharov polylogarithms [C. Duhr, H. Gangl, J. Rhodes JHEP10(2012)075]

For $n \geq 0$ and $a_i \in \mathbb{C}$ we define the *Goncharov polylogarithm* as

$$G(a_1, \dots, a_n; z) := \int_0^z \frac{dt}{t - a_1} G(a_2, \dots, a_n; t), \quad G(; z) = 1 \quad (G(0) = 0)$$

with $G(k\vec{a}; kx) = G(\vec{a}; x)$ for $a_n \neq 0, k \in \mathbb{C}^*$

Shuffle algebra: for $\Sigma(n_1, n_2) \subset S_{n_1+n_2}$ denoting shuffles

$$G(a_1, \dots, a_{n_1}; x) G(a_{n_1+1}, \dots, a_{n_1+n_2}; x) = \sum_{\sigma \in \Sigma(n_1, n_2)} G(a_{\sigma(1)}, \dots, a_{\sigma(n_1+n_2)}; x),$$

Examples:

$$\begin{aligned} G(\vec{0}_n; x) &= \frac{1}{n!} \log^n x, & G(\vec{a}_n; x) &= \frac{1}{n!} \log^n \left(1 - \frac{x}{a}\right), \\ G(\vec{0}_{n-1}, a; x) &= -\text{Li}_n \left(\frac{x}{a}\right), & G(\vec{0}_n, \vec{a}_p; x) &= (-1)^p S_{n,p} \left(\frac{x}{a}\right), \end{aligned}$$

where $S_{n,p}$ denotes the Nielsen polylogarithm.

Symbol calculus [C. Duhr, H. Gangl, J.Rhodes JHEP10(2012)075]

The differential structure

$$dG(a_{n-1}, \dots, a_1; a_n) = \sum_{i=1}^{n-1} G(a_{n-1}, \dots, \hat{a}_i, \dots, a_1; z) d \log \frac{a_i - a_{i+1}}{a_i - a_{i-1}}$$

motivates the definition of the *symbol map*

$$\mathcal{S}(G(a_{n-1}, \dots, a_1; a_n)) = \sum_{i=1}^{n-1} \mathcal{S}(G(a_{n-1}, \dots, \hat{a}_i, \dots, a_1; z)) \otimes \frac{a_i - a_{i+1}}{a_i - a_{i-1}}$$

applied iteratively it associates a tensor product (called *word*) in variables (called *letters*).

Distributivity:

$$C \otimes (a \cdot b) \otimes D = C \otimes a \otimes D + C \otimes b \otimes D$$

$$C \otimes a^n \otimes D = n(C \otimes a \otimes D), \quad n \in \mathbb{Z},$$

Shuffle product:

$$\mathcal{S}(G(a_1, \dots, a_r; x)G(b_1, \dots, b_s; y)) = \mathcal{S}(G(a_1, \dots, a_r; x)) \amalg \mathcal{S}(G(b_1, \dots, b_s; y))$$

Multiple polylogarithms [C. Duhr, H. Gangl, J. Rhodes JHEP10(2012)075]

Converging multiple polylogarithms have an expression as:

$$\text{Li}_{m_1, \dots, m_k}(x_1, \dots, x_k) = \sum_{n_1 < n_2 < \dots < n_k} \frac{x_1^{n_1} x_2^{n_2} \dots x_k^{n_k}}{n_1^{m_1} n_2^{m_2} \dots n_k^{m_k}} = \sum_{n_k=1}^{\infty} \frac{x_k^{n_k}}{n_k^{m_k}} \sum_{n_{k-1}=1}^{n_k-1} \dots \sum_{n_1=1}^{n_{k-1}-1} \frac{x_1^{n_1}}{n_1^{m_1}}.$$

They are related to Goncharov polylogarithms via

$$\text{Li}_{m_1, \dots, m_k}(x_1, \dots, x_k) = (-1)^k G_{m_k, \dots, m_1} \left(\frac{1}{x_k}, \dots, \frac{1}{x_1 \dots x_k} \right)$$

Example 0:

$$\text{Li}_k(z) = \int_0^z \text{Li}_{k-1}(t) d \log t, \quad \text{Li}_1(z) = -\log(1-z) \quad (15)$$

clearly have

$$\text{symbol}(\text{Li}_k(z)) = -(1-z) \otimes \underbrace{z \otimes \dots \otimes z}_{k-1 \text{ times}}. \quad (16)$$

Example 1: $G(a, b; x)$ [C. Duhr, H. Gangl, J.Rhodes JHEP10(2012)075]

Example 1: For a, b distinct and non zero one verifies

$$G(a, b; x) = \text{Li}_2\left(\frac{b-x}{b-a}\right) - \text{Li}_2\left(\frac{b}{b-a}\right) + \log\left(1 - \frac{x}{b}\right) \log\left(\frac{x-a}{b-a}\right)$$

Combined with **Example 0** we may deduce:

$$\mathcal{S}(G(a, b; x)) = \left(1 - \frac{x}{b}\right) \otimes \left(1 - \frac{x}{a}\right) - \left(1 - \frac{x}{b}\right) \otimes \left(1 - \frac{b}{a}\right) + \left(1 - \frac{x}{a}\right) \otimes \left(1 - \frac{a}{b}\right)$$

Example 2: Scattering amplitude

[A. Goncharov, M. Spradlin, C. Vergu, A. Volovich. Phys. Rev. Lett. 105 (2010) 151605]

After manipulating the ($\mathcal{N} = 4$ super Yang-Mills six particle two loop) *scattering amplitude* we are left with the remainder function:

$$R_6^{(2)} = \sum_{i=1}^3 \left(L_i - \frac{1}{2} \text{Li}_4(-v_i) \right) - \frac{1}{8} \left(\sum_{i=1}^3 \text{Li}_2(-v_i) \right)^2 + \frac{1}{24} J^4 + \frac{\pi^2}{12} J^2 + \frac{\pi^4}{72},$$

in terms of the functions

$$L_i = \frac{1}{384} P_i^4 + \sum_{m=0}^3 \frac{(-1)^m}{(2m)!!} P_i^m (\ell_{4-m}(x_i^+) + \ell_{4-m}(x_i^-)),$$

$$P_i = 2 \text{Li}_1(-v_i) - \sum_{j=1}^3 \text{Li}_1(-v_j),$$

and

$$J = \sum_{i=1}^3 \ell_1(x_i^+) - \ell_1(x_i^-),$$

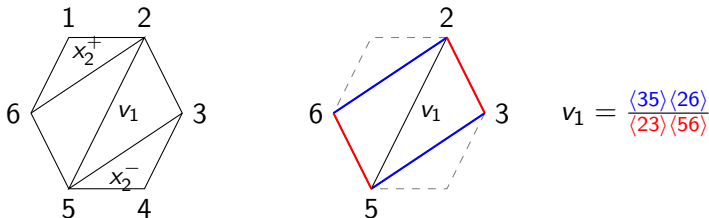
$$\ell_n(x) = \frac{1}{n} (\text{Li}_n(-x) - (-1)^n \text{Li}_n(-1/x)).$$

Example 2: symbol of a scattering amplitude

[A. Goncharov, M. Spradlin, C. Vergu, A. Volovich. Phys. Rev. Lett. 105 (2010) 151605]

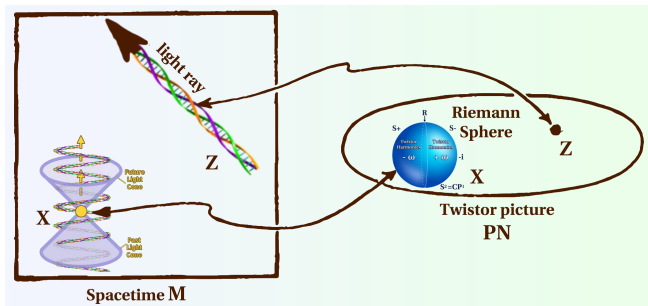
$$\begin{aligned}
 v_1 &= \frac{\langle 35 \rangle \langle 26 \rangle}{\langle 23 \rangle \langle 56 \rangle}, & v_2 &= \frac{\langle 13 \rangle \langle 46 \rangle}{\langle 16 \rangle \langle 34 \rangle}, & v_3 &= \frac{\langle 15 \rangle \langle 24 \rangle}{\langle 45 \rangle \langle 12 \rangle}, \\
 x_1^+ &= \frac{\langle 14 \rangle \langle 23 \rangle}{\langle 12 \rangle \langle 34 \rangle}, & x_2^+ &= \frac{\langle 25 \rangle \langle 16 \rangle}{\langle 56 \rangle \langle 12 \rangle}, & x_3^+ &= \frac{\langle 36 \rangle \langle 45 \rangle}{\langle 34 \rangle \langle 56 \rangle}, \\
 x_1^- &= \frac{\langle 14 \rangle \langle 56 \rangle}{\langle 45 \rangle \langle 16 \rangle}, & x_2^- &= \frac{\langle 25 \rangle \langle 34 \rangle}{\langle 23 \rangle \langle 45 \rangle}, & x_3^- &= \frac{\langle 36 \rangle \langle 12 \rangle}{\langle 16 \rangle \langle 23 \rangle}
 \end{aligned}$$

Observe that combinatorially the symbol alphabet corresponds to *quadrilaterals in a hexagon*:



Twistor theory

[R. Penrose. *Twistor algebra*. J. Math. Phys. 8, 345–366 (1967)]



©<https://twistor.li/>

Momentum twistors

[A. Hodges. *Eliminating spurious poles from gauge-theoretic amplitudes* JHEP, 2013(5), 135]

A particle configuration in $\mathcal{N} = 4$ super Yang Mills may be represented by momentum twistors $Z_1, \dots, Z_n \in \mathbb{CP}^3$. The system is parametrized by

$$\langle ijkl \rangle := \det(Z_i Z_j Z_k Z_l),$$

for $1 \leq i < j < k < l \leq n$ satisfying determinantal identities (Plücker relations). This configuration space is closely related to the **Grassmannian** $\text{Gr}_{4,n}$ (up to scaling).

Example 2: For $n = 6$ we have $\text{Gr}_{4,6} \cong \text{Gr}_{2,6}$ and the symbol alphabet of $R_6^{(2)}$ consists of the $\mathcal{X}$ cluster variables (resp. $\hat{y}$ -variables).

Symbol alphabet and cluster algebras

In $\mathcal{N} = 4$ super Yang–Mills the symbol alphabet of the scattering amplitude for n particles

$n \leq 7$ $\text{Gr}_{4,n}$ has finitely many cluster variables and they determine the complete symbol alphabet

[Golden, Goncharov, Spradlin, Vergu, Volovich. JHEP 01 (2014) 091]

$n = 8$ $\text{Gr}_{4,8}$ has infinitely many cluster variables, however the complete alphabet of 272 letters is determined by a finite collection of cluster variables and expressions obtained as limits along infinite mutation sequences of Kronecker type.

[Drummond, Foster, Gurdogan, Kalousios. JHEP 11 (2021) 071]

Criticism: $\mathcal{N} = 4$ SYM is a toy model *too far* from the real world modeled by the standard model

Question: Are cluster algebras relevant in other models?

Beyond $\mathcal{N} = 4$ super Yang–Mills

Introduce an *infinity twistor*, a line spanned by $Z_{n+1}, Z_{n+2} \in \mathbb{CP}^3$, breaking cyclic symmetry in the model and parametrise by

$$\langle ijkl \rangle := \det(Z_i Z_j Z_k Z_l), \quad \langle ij \rangle := \det(Z_i Z_j Z_{n+1} Z_{n+2}),$$

for $1 \leq i < j < k < l \leq n$ resp. $1 \leq i < j \leq n$.

This motivates us to define the *momentum twistor variety* as the subvariety of $\mathbb{P}^{\binom{n}{2}-1} \times \mathbb{P}^{\binom{n}{4}-1}$ given by the vanishing of determinantal identities among the $\langle ijkl \rangle$ and $\langle ij \rangle$.

The momentum twistor variety $\mathcal{MT}_n$ is the *partial flag variety* $\mathcal{F}_{2,4;n}$.

Beyond $\mathcal{N} = 4$ super Yang–Mills

Let $\{p_1, \dots, p_n\} \subset \mathbb{R}^4$ be a configuration of n lightlike particles in QCD (standard model)

- 1 its helicity spinors are n pairs $(\lambda_i, \tilde{\lambda}_i) \in \mathbb{C}^{2 \times 2}$ with $\lambda_i \tilde{\lambda}_i^T = 0$
- 2 define $\Lambda = [\lambda_1, \dots, \lambda_n], \tilde{\Lambda} = [\tilde{\lambda}_1, \dots, \tilde{\lambda}_n] \in \text{Gr}_{2;n}$ with $\Lambda \tilde{\Lambda}^T = 0$

The *spinor helicity variety*

$$\mathcal{SH}_n := \{(\Lambda, \tilde{\Lambda}) \in \text{Gr}_{2;n} \times \text{Gr}_{2;n} : \Lambda \tilde{\Lambda}^T = 0\}.$$

is parametrized by 2×2 minors with $1 \leq i < j \leq n$

$$\langle ij \rangle := \det(\lambda_i \lambda_j) \quad \text{y} \quad [ij] := \det(\tilde{\lambda}_i \tilde{\lambda}_j)$$

satisfying *Plücker relations* and *momentum conservation*

$$0 = \langle ij \rangle \langle kl \rangle - \langle ik \rangle \langle jl \rangle + \langle il \rangle \langle jk \rangle = [ij][kl] - [ik][jl] + [il][jk]$$

$$0 = \sum_{s=1}^n \langle is \rangle [sj] \quad (\Leftrightarrow \Lambda \tilde{\Lambda}^T = 0)$$

Spinor helicity variety

[Y.El Mazzouz, A.Pfister, and B.Sturmfels. 2406.17331]

Consider the map

$$\langle ij \rangle \mapsto P_{ij}, \quad \text{and} \quad [ij] \mapsto (-1)^{i+j-1} P_{[n]-ij}$$

where $[n] - ij := \{1, \dots, n\} - \{i, j\}$. Momentum conservation also turns into Plücker relations

$$\begin{aligned} P_{ij}P_{kl} - P_{ik}P_{jl} + P_{il}P_{jk} &= 0 \\ P_{[n]-ij}P_{[n]-kl} - P_{[n]-ik}P_{[n]-jl} + P_{[n]-il}P_{[n]-jk} &= 0 \\ \sum_{s=1}^n (-1)^{s+j-1} P_{is}P_{[n]-js} &= 0 \end{aligned}$$

As $\dim \mathcal{SH}_n = 4(n-3) = \dim \mathcal{F}_{2,n-2;n}$ the map induces an isomorphism between the spinor helicity variety $\mathcal{SH}_n$ and the *partial flag variety* $\mathcal{F}_{2,n-2;n}$.

Cluster algebras

Theorem (C. Geiss, B. Leclerc, J. Schröer. Ann. Inst. Fourier 2008)

The homogeneous coordinate ring of a partial flag variety in its Plücker embedding $\mathbb{C}[\mathcal{F}_{d_1, \dots, d_k; n}]$ is a cluster algebra.

Both, the spinor helicity varieties by $\mathcal{F}_{2, n-2; n}$ and the momentum twistor varieties by $\mathcal{F}_{2, 4; n}$ inherit these cluster structures.

Symbol alphabet for 5 particles in QCD

[L.B., J. Drummond, R. Glew. JHEP11(2023)002]

Analytic computations have found a symbol alphabet of 31 letters, one of which has unclear physical interpretation

[T. Gehrmann, J. M. Henn and N. A. Lo Presti. Phys. Rev. Lett. 116 (2016) 062001]

[D. Chicherin, J. Henn and V. Mitev. JHEP 05 (2018) 164]

Reformulated in spinor helicity variables up to cyclic symmetry it is

$$W_1 = \langle 12 \rangle [12],$$

$$W_6 = \langle 34 \rangle [34] + \langle 45 \rangle [45],$$

$$W_{11} = \langle 34 \rangle [34] + \langle 35 \rangle [35],$$

$$W_{16} = \langle 13 \rangle [13],$$

$$W_{21} = \langle 13 \rangle [13] + \langle 34 \rangle [34],$$

$$W_{26} = \frac{\langle 45 \rangle [51] \langle 12 \rangle [24]}{[45] \langle 51 \rangle [12] \langle 24 \rangle},$$

$$W_{31} = [45] \langle 51 \rangle [12] \langle 24 \rangle - \langle 45 \rangle [51] \langle 12 \rangle [24].$$

$$W_6 = \langle 34 \rangle [34] + \langle 45 \rangle [45] = \langle 12 \rangle [12] - \langle 35 \rangle [35],$$

$$W_{11} = \langle 34 \rangle [34] + \langle 35 \rangle [35] = \langle 12 \rangle [12] - \langle 45 \rangle [45],$$

$$W_{21} = \langle 13 \rangle [13] + \langle 34 \rangle [34] = \langle 14 \rangle [14] - \langle 25 \rangle [25],$$

Symbol alphabet for 5 particles in QCD

[L.B., J. Drummond, R. Glew. JHEP11(2023)002]

$\mathcal{F}_{2,3;5}$ is of finite cluster type D_4 and has 16 cluster variables and 6 frozen variables.

- letters $W_1, \dots, W_5, W_{16}, \dots, W_{20}, W_{26}, \dots, W_{30}$ are multiplicative combinations of Plücker coordinates (cluster variables of degree one)
- letters $W_6, \dots, W_{10}, W_{11}, \dots, W_{15}, W_{21}, \dots, W_{25}$ are S_5 permutation copies of cluster variables of degree two
- spurious letter W_{31} is not recovered by the cluster algebra

Embeddings of cluster algebras

Let Q be a quiver and let Q' be obtained from Q by freezing and deleting vertices. Then Q' is called a *restricted quiver of Q* .

→ get induced notion of *restricted seeds* determining *cluster subpatterns* and *cluster subalgebras*.

Let p_J with $J \subset [n]$ with $|J| \in \{d_1, \dots, d_k\}$ be Plücker coordinates of $\mathbb{C}[\mathcal{F}_{d_1, \dots, d_k; n}]$, and P_J with $|J| = d_k$ be the Plücker coordinates of $\mathbb{C}[\text{Gr}_{d_k, N}]$, $N = n + d_k - d_1$. Define for $|J| = d_j$:

$$p_J \mapsto P_{J \cup [n+1, \dots, n+d_k-d_j]}$$

Theorem (L.B., Jianrong Li. arxiv:2408.14956)

The above map extends to an embedding of cluster algebras

$$\mathbb{C}[\mathcal{F}_{d_1, \dots, d_k; n}] \hookrightarrow \mathbb{C}[\text{Gr}_{d_k; N}]$$

Symbol alphabet for 6 particles in QCD

[A. Pokraka, M. Spradlin, A. Volovicha, H.-C. Weng, arxiv:2506.11895]

L.B., J. Drummond, E. Glew, Ö. Gürdoğan, and R. Wright. arxiv:2507.01015

Analytic computations have found a symbol alphabet of *289 letters* that split into *38 permutation classes*

[J. Henn, A. Matijašić, J. Miczajka, T. Peraro, et al. Phys.Rev.Lett. 135 (2025) 3, 031601]

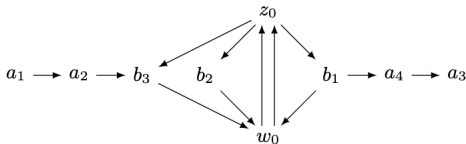
$\mathcal{F}_{2,4;6}$ is of affine type $D_6^{(1)}$, so it has infinitely many cluster variables

- we identify 54 cluster variables that are letters and give 11 permutation classes
- 32 letters are obtained as limits of infinite mutation sequences along a Kronecker subquiver from 16 different seeds
- Using the embedding $\mathbb{C}[\mathcal{F}_{2,4;6}] \hookrightarrow \mathbb{C}[\text{Gr}_{4,8}]$ and known results for $\text{Gr}_{4,8}$ we recover a total of 32 out of the 38 permutation classes

Infinite mutation sequences

[L.B., J. Drummond, E. Glew, Ö. Gürdoğan, and R. Wright. [arxiv:2507.01015](https://arxiv.org/abs/2507.01015)]

A seed for $\text{Gr}_{4,8}$ (without frozen) that restricts to a seed for $\mathcal{F}_{2,4;6}$:



$$z_0 = p_{1236} p_{1578} - p_{1235} p_{1678}$$

$$w_0 = p_{1356}$$

$$b_1 = p_{1236} p_{3578} - p_{1235} p_{3678}$$

$$a_1 = p_{1345}$$

$$b_2 = p_{1256}$$

$$a_2 = p_{1346}$$

$$b_3 = p_{1346} p_{1578} - p_{1345} p_{1678}$$

$$a_3 = p_{1237}$$

$$a_4 = p_{1236} p_{4578} - p_{1235} p_{4678} + p_{1234} p_{5678}$$

Mutate w_0 and z_0 , get a sequence z_i : $z_{i+2}z_i = b_1b_2b_3F + z_{i+1}^2$.

Solve for z_i , get an expression containing the letter $\sqrt{\Delta}$ with

$$\begin{aligned} \Delta = & p_{12}^2 p_{3456}^2 - 2p_{1256} p_{12} p_{3456} p_{34} + p_{1256}^2 p_{34}^2 \\ & - 2p_{1234} p_{12} p_{3456} p_{56} - 2p_{1234} p_{1256} p_{3478} p_{56} + p_{1234}^2 p_{56}^2 \end{aligned}$$



ICMAM Latin America

International
Community of **Mathematicians**
from **Latin America**

The international community of mathematicians ICMAM now has its own Springer book series, the first addition on Algebra, Analysis and Logic:

