Cluster Algebras

Lara Bossinger

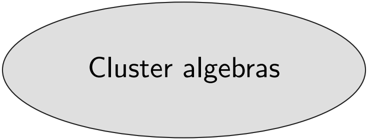
Universidad Nacional Autónoma de México, Unidad Oaxaca

Pittsburgh September 25 2020, 55 minutes

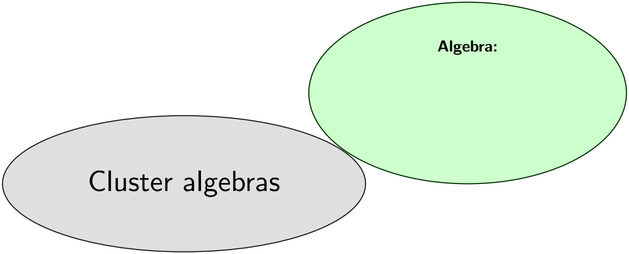
Overview

1. Cluster algebras in Mathematics
2. History
3. Total positivity
4. Quivers, seeds, mutation and cluster algebras
5. Some applications

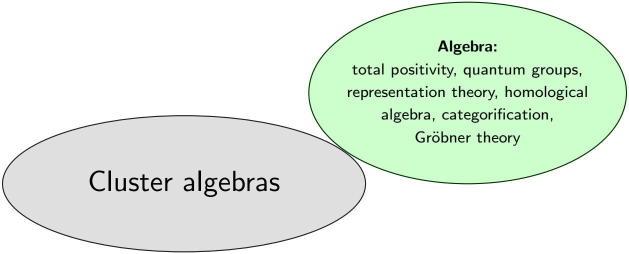
Cluster algebras in Mathematics



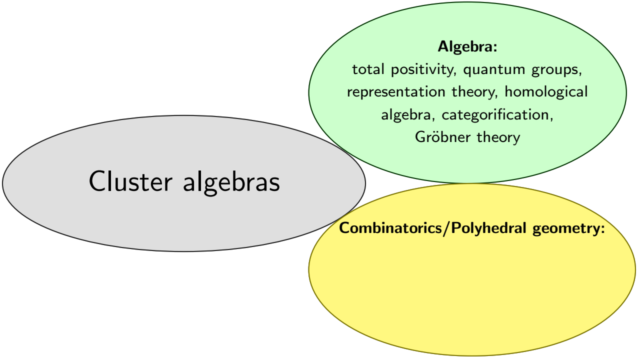
Cluster algebras in Mathematics



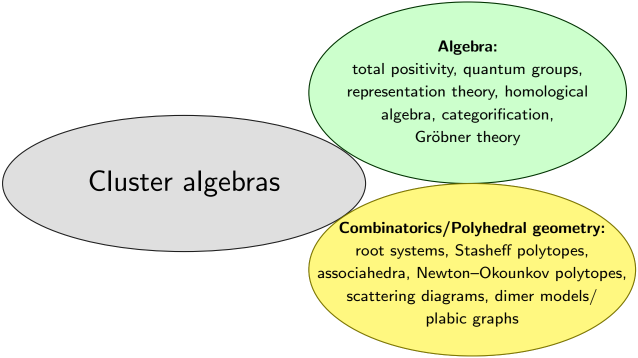
Cluster algebras in Mathematics



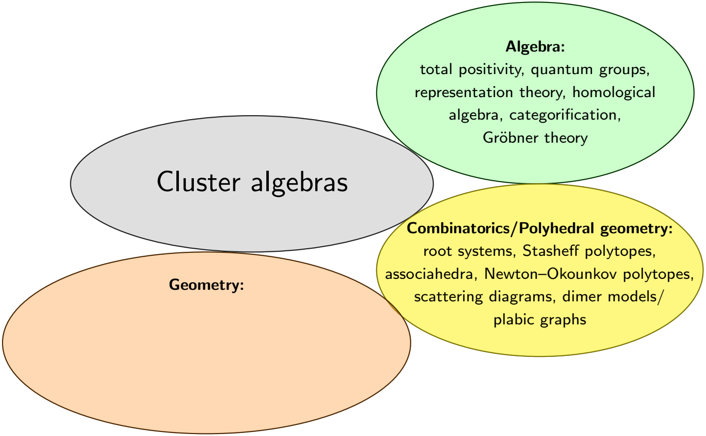
Cluster algebras in Mathematics



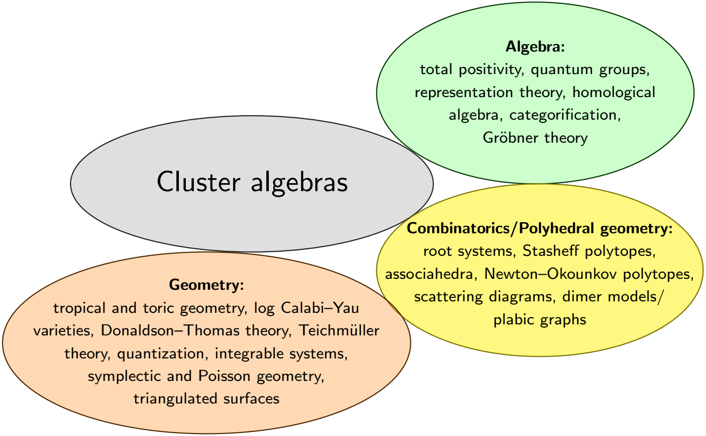
Cluster algebras in Mathematics



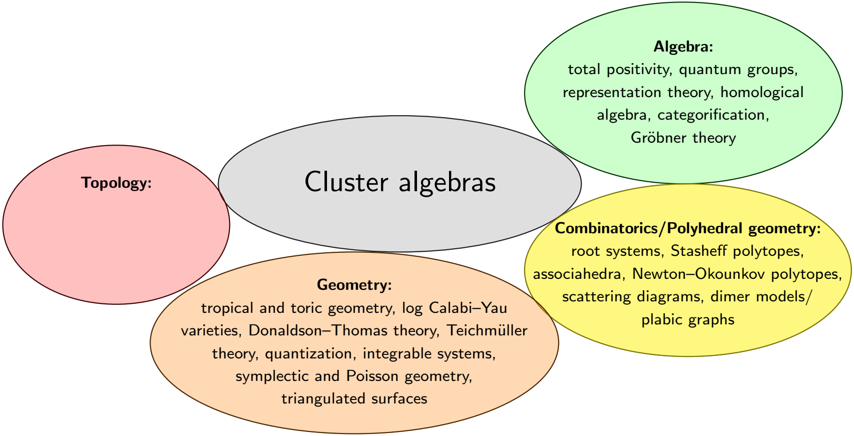
Cluster algebras in Mathematics



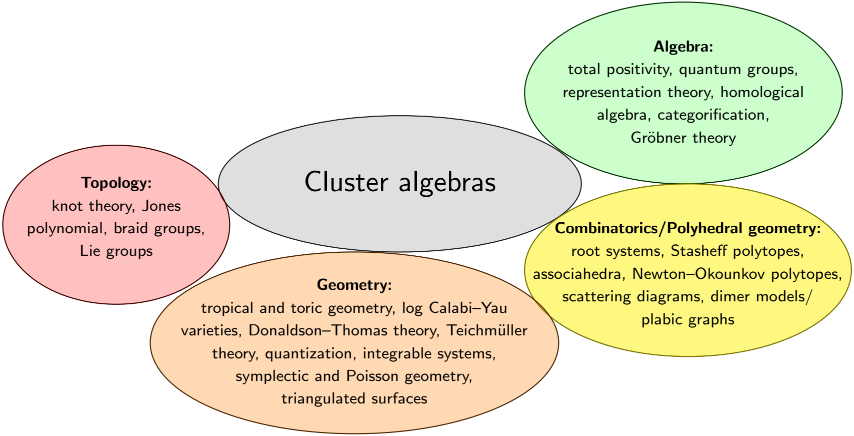
Cluster algebras in Mathematics



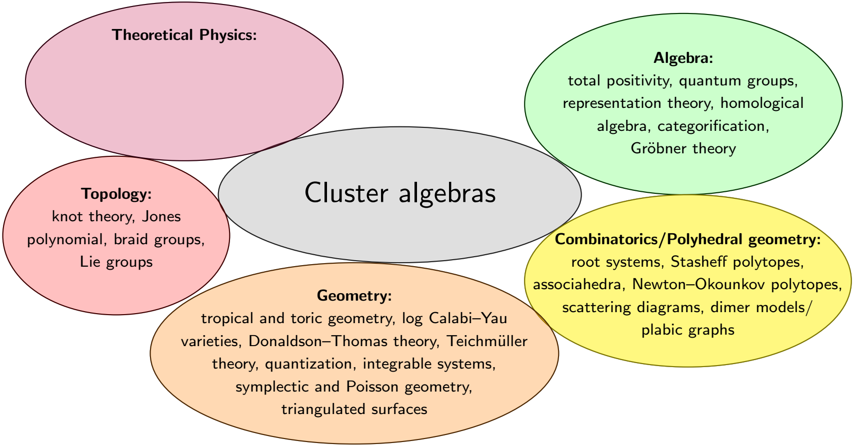
Cluster algebras in Mathematics



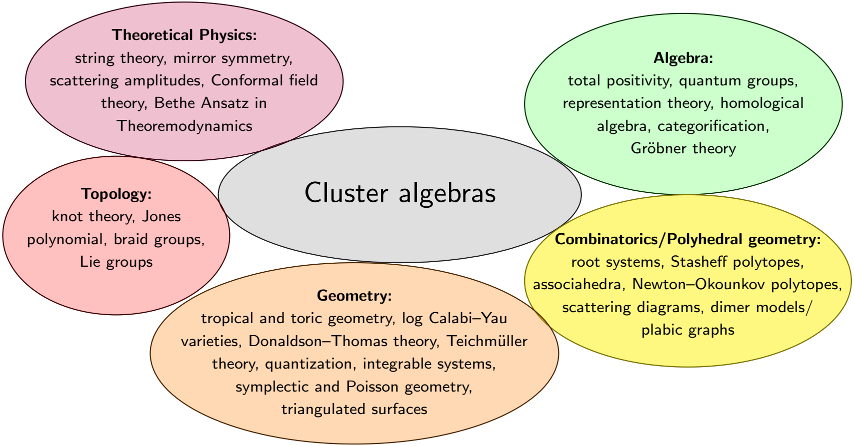
Cluster algebras in Mathematics



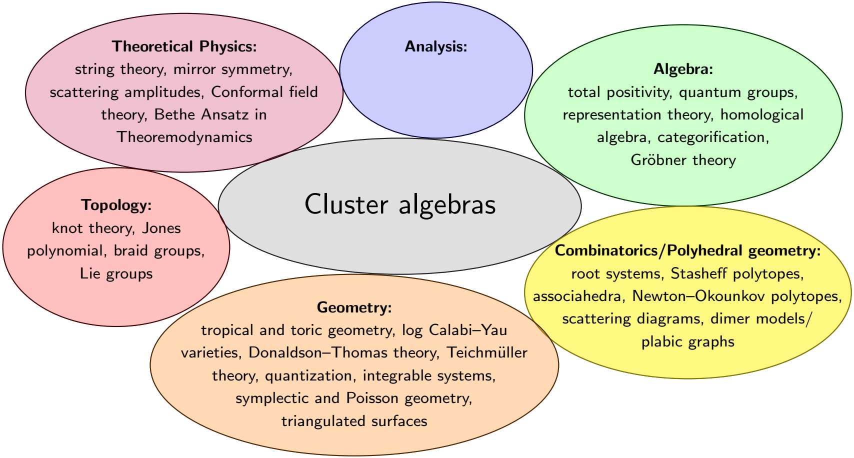
Cluster algebras in Mathematics



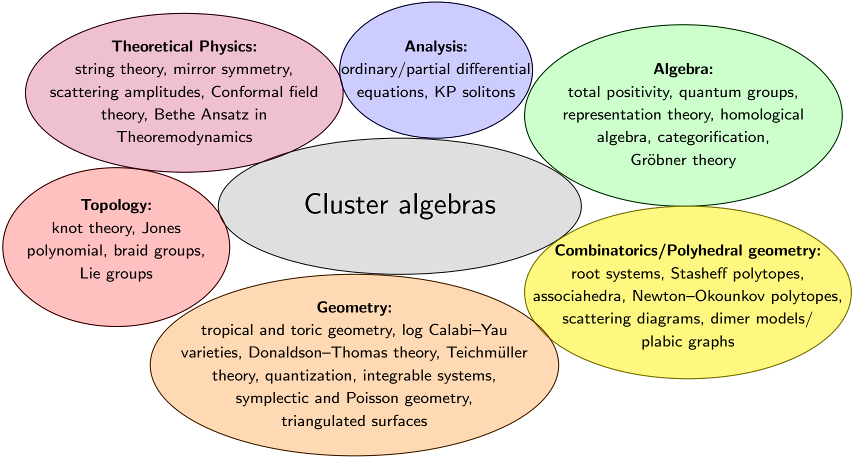
Cluster algebras in Mathematics



Cluster algebras in Mathematics



Cluster algebras in Mathematics



History

- Cluster algebras were first introduced in 2001 by Sergei Fomin and Andrei Zelevinsky.
- They had observed a mathmatical structure (*cluster structure*) on various objects related to the study of totally positive matrices, quantum groups and Kashiwara/Lusztig's canonical basis.
- Their work gained a lot of attention, first in representation theory, but quickly grew beyond its origins.
- Today Cluster algebras have their own MSC classifier 13F60, there are 690 papers on Mathscinet with this MSC classification and 1,645 on the arxiv.
- Since 2003 at least 139 international conferences have been organized on this topic

Total Positivity

Definition

A matrix $M \in \mathbb{R}^{n\times n}$ is *totally positive* if all its minors (i.e. determinantes of sub-squarematrices) are positive real numbers.

Example

Take $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then M is totally positive if and only if a, b, c, d and $\Delta = ad - cd \in \mathbb{R}_{>0}$.
Observe that $d = \frac{1}{a}(\Delta + cd)$. Hence, it suffices to verify that $a, b, c, \Delta \in \mathbb{R}_{>0}$. The set $\{a, b, c, \Delta\}$ is a *positivity test*.

Question: How can we efficiently test for total positivity?

Totally positive Grassmannian $\mathrm{Gr}_2(\mathbb{C}^n)_{>0}$

$\mathrm{Gr}_2(\mathbb{C}^n) := \{V \subset \mathbb{C}^n \mid \dim V = 2\}$ and its elements can be represented by matrices: for $V = \langle v, w \rangle$ define $M_V = \begin{bmatrix} v_1 & v_2 & \dots & v_n \\ w_1 & w_2 & \dots & w_n \end{bmatrix} \in \mathbb{C}^{2\times n}$ (unique up to rescaling of the rows).

Definition

For $i, j \in \{1, 2, \dots, n\}$ with $i < j$ define for $V \in \mathrm{Gr}_2(\mathbb{C}^n)$ with M_V as above

$$p_{ij}(V) := \det \begin{bmatrix} v_i & v_j \\ w_i & w_j \end{bmatrix}.$$

The *totally positive Grassmannian* $\mathrm{Gr}_2(\mathbb{C}^n)_{>0}$ consist of those points $V \in \mathrm{Gr}_2(\mathbb{C}^n)$ for which $p_{ij}(V) \in \mathbb{R}_{>0}$ for all i, j.

Question: There are $\binom{n}{2}$ Plücker coordinates. How many do we have to test to know that a given point lies in $\mathrm{Gr}_2(\mathbb{C}^n)_{>0}$?

Ptolemy and Plücker relations

Plücker coordinates are not independent.

Ptolemy and Plücker relations

Plücker coordinates are not independent. One verifies that for $i < j < k < l \in \{1, \ldots, n\} : p_{ik}p_{jl} = p_{ij}p_{kl} + p_{il}p_{jk}$.

Ptolemy and Plücker relations

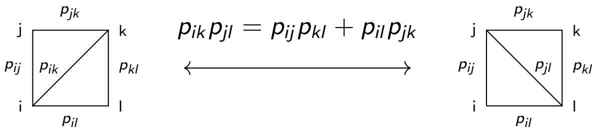
Plücker coordinates are not independent. One verifies that for $i < j < k < l \in \{1, \ldots, n\} : p_{ik}p_{jl} = p_{ij}p_{kl} + p_{il}p_{jk}$.

Can vizualize the *Plücker relation*:

Ptolemy and Plücker relations

Plücker coordinates are not independent. One verifies that for $i < j < k < l \in \{1, \ldots, n\} : p_{ik}p_{jl} = p_{ij}p_{kl} + p_{il}p_{jk}$.

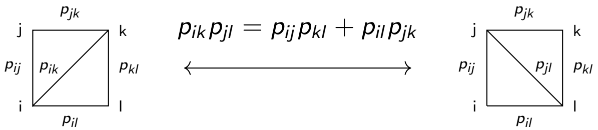
Can vizualize the *Plücker relation*:



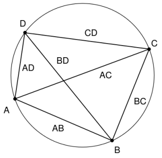
Ptolemy and Plücker relations

Plücker coordinates are not independent. One verifies that for $i < j < k < l \in \{1, \ldots, n\} : p_{ik}p_{jl} = p_{ij}p_{kl} + p_{il}p_{jk}$.

Can vizualize the *Plücker relation*:



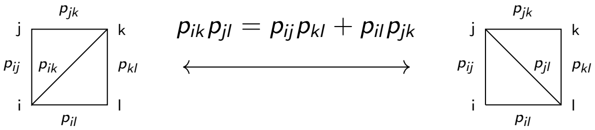
$\rightarrow$ compare to the *Ptolemy relation*:



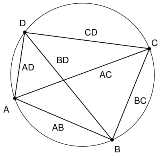
Ptolemy and Plücker relations

Plücker coordinates are not independent. One verifies that for $i < j < k < l \in \{1, \ldots, n\}$: $p_{ik}p_{jl} = p_{ij}p_{kl} + p_{il}p_{jk}$.

Can vizualize the *Plücker relation*:



$\rightarrow$ compare to the *Ptolemy relation*: $\overline{AC} \cdot \overline{BD} = \overline{AB} \cdot \overline{CD} + \overline{BC} \cdot \overline{AD}$



TP tests for $\mathrm{Gr}_2(\mathbb{C}^n)$

Consequence: $\left\{\begin{matrix}\text{efficient TP} \\ \text{tests for } \mathrm{Gr}_2(\mathbb{C}^n)\end{matrix}\right\} \xleftrightarrow{1-1} \left\{\begin{matrix}\text{triangulations} \\ \text{of an } n\text{-gon}\end{matrix}\right\}.$

TP tests for $\mathrm{Gr}_2(\mathbb{C}^n)$

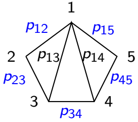
Consequence: $\left\{\begin{matrix}\text{efficient TP}\\ \text{tests for } \mathrm{Gr}_2(\mathbb{C}^n)\end{matrix}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{matrix}\text{triangulations}\\ \text{of an } n\text{-gon}\end{matrix}\right\}$.

$\rightarrow$ it suffices to check a set of $2(n-2)+1 = \dim \mathrm{Gr}_2(\mathbb{C}^n) + 1$ independent Plücker coordinates for positivity.

TP tests for $\mathrm{Gr}_2(\mathbb{C}^n)$

Consequence: $\left\{\begin{matrix}\text{efficient TP}\\ \text{tests for } \mathrm{Gr}_2(\mathbb{C}^n)\end{matrix}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{matrix}\text{triangulations}\\ \text{of an } n\text{-gon}\end{matrix}\right\}.$

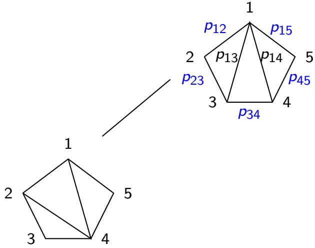
$\rightarrow$ it suffices to check a set of $2(n-2)+1 = \dim \mathrm{Gr}_2(\mathbb{C}^n)+1$ independent Plücker coordinates for positivity.



TP tests for $\mathrm{Gr}_2(\mathbb{C}^n)$

Consequence: $\left\{ \begin{matrix} \text{efficient TP} \\ \text{tests for } \mathrm{Gr}_2(\mathbb{C}^n) \end{matrix} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{matrix} \text{triangulations} \\ \text{of an } n\text{-gon} \end{matrix} \right\}.$

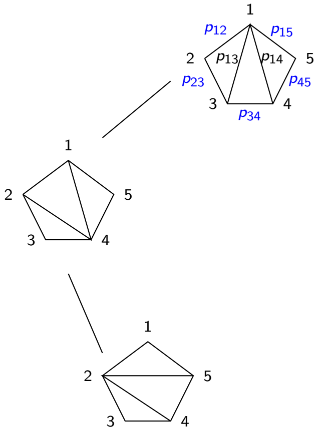
$\rightarrow$ it suffices to check a set of $2(n-2)+1 = \dim \mathrm{Gr}_2(\mathbb{C}^n) + 1$ independent Plücker coordinates for positivity.



TP tests for $\mathrm{Gr}_2(\mathbb{C}^n)$

Consequence: $\left\{\begin{matrix}\text{efficient TP}\\ \text{tests for } \mathrm{Gr}_2(\mathbb{C}^n)\end{matrix}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{matrix}\text{triangulations}\\ \text{of an } n\text{-gon}\end{matrix}\right\}$.

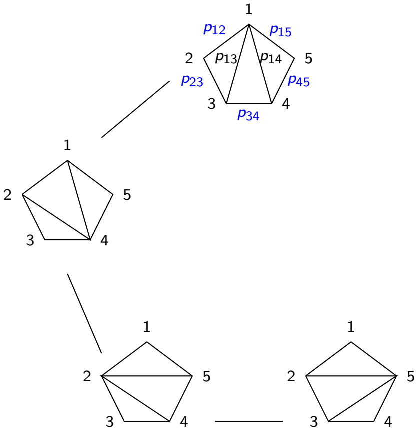
$\rightarrow$ it suffices to check a set of $2(n-2)+1 = \dim \mathrm{Gr}_2(\mathbb{C}^n) + 1$ independent Plücker coordinates for positivity.



TP tests for $\mathrm{Gr}_2(\mathbb{C}^n)$

Consequence: $\left\{ \begin{matrix} \text{efficient TP} \\ \text{tests for } \mathrm{Gr}_2(\mathbb{C}^n) \end{matrix} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{matrix} \text{triangulations} \\ \text{of an } n\text{-gon} \end{matrix} \right\}$.

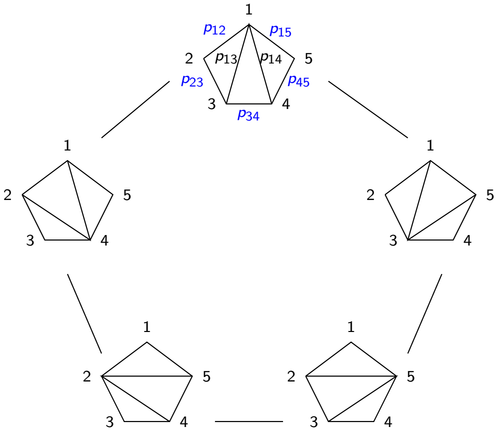
$\rightarrow$ it suffices to check a set of $2(n-2)+1 = \dim \mathrm{Gr}_2(\mathbb{C}^n) + 1$ independent Plücker coordinates for positivity.



TP tests for $\mathrm{Gr}_2(\mathbb{C}^n)$

Consequence: $\left\{ \begin{array}{c} \text{efficient TP} \\ \text{tests for } \mathrm{Gr}_2(\mathbb{C}^n) \end{array} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{triangulations} \\ \text{of an } n\text{-gon} \end{array} \right\}.$

$\rightarrow$ it suffices to check a set of $2(n-2)+1 = \dim \mathrm{Gr}_2(\mathbb{C}^n) + 1$ independent Plücker coordinates for positivity.



Quivers and mutation

A *quiver* Q is a directed graph, consisting of a finite set of vertices and arrows between them.

Technical assumption: Q does not have any loops or 2-cycles.

We split the vertex set into *mutable vertices* $\{1, \dots, n\}$ and *frozen vertices* $\{n+1, \dots, m\}$, e.g. $1 \rightrightarrows 2 \to 3$.

Definition (Quiver mutation)

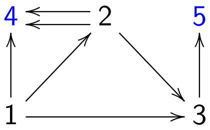
Given a quiver Q and a mutable vertex k, the *mutation in direction k* $\mu_k(Q)$ is a quiver obtained from Q in three steps:

1. for every path $i \to k \to j$ add an arrow $i \to j$;
2. invert every arrow incident to k;
3. remove a maximal set of 2-cycles.

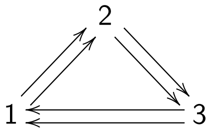
$\to$ Quiver mutation is an *involution*: $\mu_k(\mu_k(Q)) = Q$.

Example: quiver mutation in Keller's mutation app

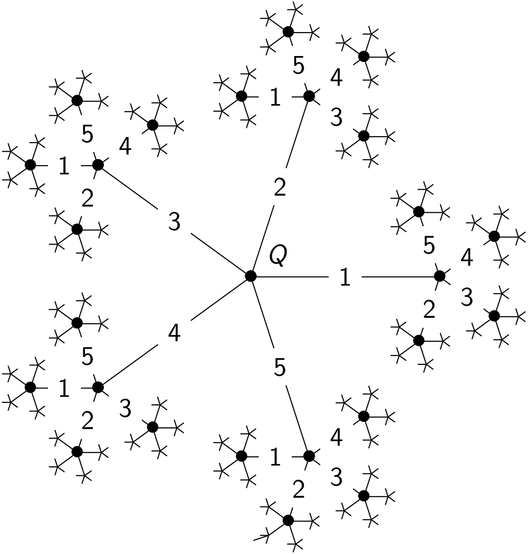
1. Q_1 :



2. Q_2 :



Iteration of mutations: the n-regular tree



Seeds and mutation

A *seed* s is a pair $(\mathbf{x}, Q)$, where $\mathbf{x} = (x_1, \dots, x_n, x_{n+1}, \dots, x_{n+m})$ is a collection of variables called a *cluster* and Q a quiver with n mutable and m frozen vertices.

Definition (Seed mutation)

Given a seed $s = (\mathbf{x}, Q)$ and a mutable vertex k of Q, the *mutation in direction k* $\mu_k(s)$ is the pair $(\mu_k(\mathbf{x}), \mu_k(Q))$, where $\mu_k(\mathbf{x}) = \mathbf{x} \setminus \{x_k\} \cup \{x_k'\}$ and

$$x_k' := \frac{\prod_{i \to k \in Q} x_i + \prod_{k \to j \in Q} x_j}{x_k}.$$

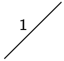
$\rightarrow$ Seed mutation is an *involution*: $\mu_k(\mu_k(s)) = s$.

Example: seed mutation

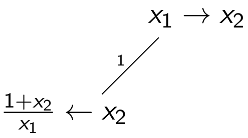
$$x_1 \rightarrow x_2$$

Example: seed mutation

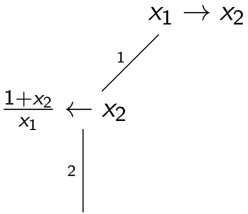
$$x_1 \to x_2$$



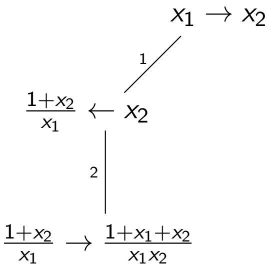
Example: seed mutation



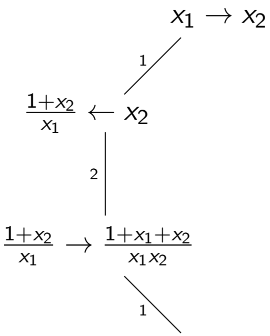
Example: seed mutation



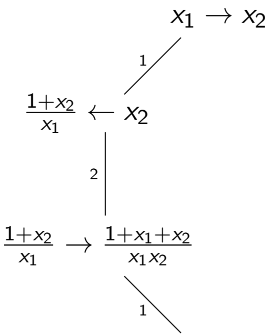
Example: seed mutation



Example: seed mutation

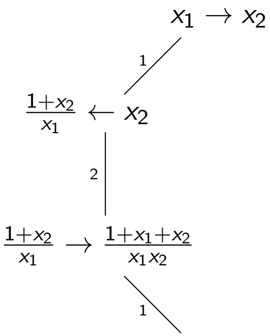


Example: seed mutation



$$\left(1 + \frac{1 + x_1 + x_2}{x_1 x_2}\right) \div \frac{1 + x_2}{x_1}$$

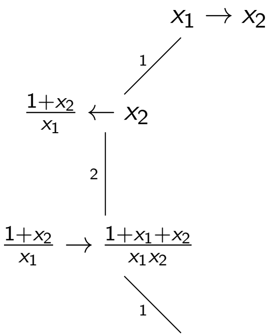
Example: seed mutation



cluster magic!

$$\left(1 + \frac{1 + x_1 + x_2}{x_1 x_2}\right) \div \frac{1 + x_2}{x_1} \quad = \frac{x_1 x_2 + 1 + x_1 + x_2}{x_2(1 + x_2)}$$

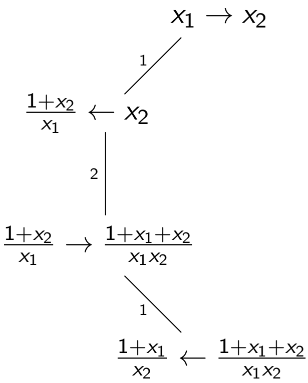
Example: seed mutation



cluster magic!

$$\left(1 + \frac{1 + x_1 + x_2}{x_1 x_2}\right) \div \frac{1 + x_2}{x_1} \quad = \frac{x_1 x_2 + 1 + x_1 + x_2}{x_2(1 + x_2)} \quad = \frac{1 + x_1}{x_2}$$

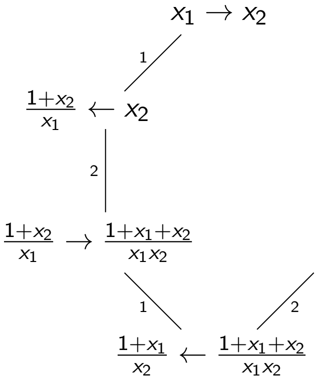
Example: seed mutation



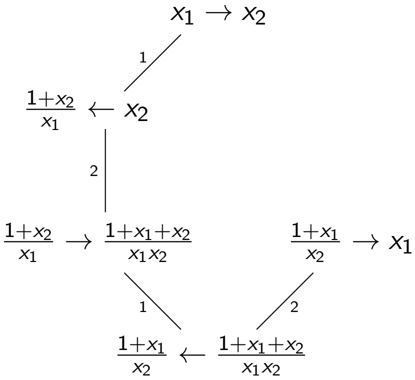
cluster magic!

$$\left(1+\frac{1+x_1+x_2}{x_1x_2}\right) \div \frac{1+x_2}{x_1} = \frac{x_1x_2+1+x_1+x_2}{x_2(1+x_2)} = \frac{1+x_1}{x_2}$$

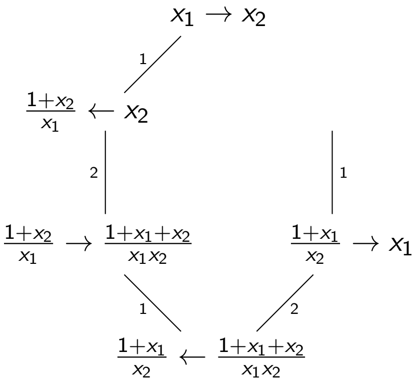
Example: seed mutation



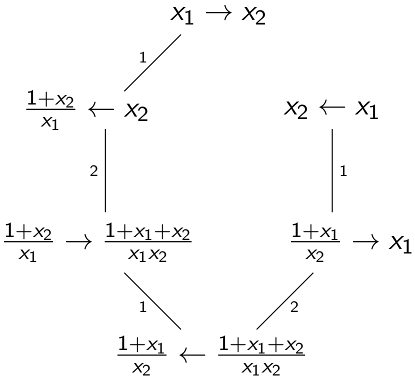
Example: seed mutation



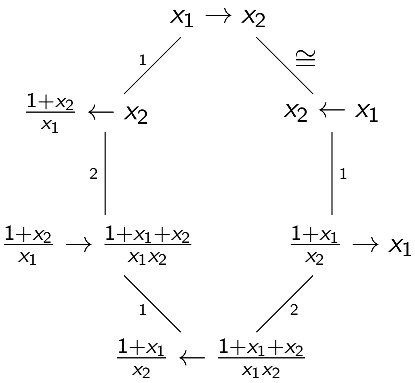
Example: seed mutation



Example: seed mutation



Example: seed mutation



Cluster algebra

Let $\mathcal{F} = \mathbb{C}(x_1, \ldots, x_{n+m})$ be the field of rational functions in the variables $x_1, \ldots, x_{n+m}$.

If $s = (\mathbf{x}, Q)$ is a seed with $\mathbf{x} = (x_1, \ldots, x_{n+m})$ and let $s' = (\mathbf{x}', Q')$ be a seed obtained from s by a sequence of mutations, then the cluster $\mathbf{x}' = (x'_1, \ldots, x'_n, x'_{n+1}, \ldots, x'_{n+m})$ satisfies

$$\mathbb{C}(x'_1, \ldots, x'_n, x'_{n+1}, \ldots, x'_{n+m}) = \mathcal{F}.$$

Definition

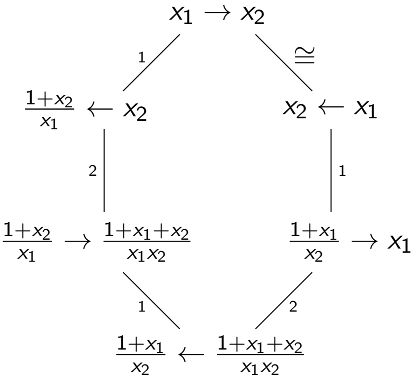
The *cluster algebra* defined by the initial quiver Q is the $\mathcal{F}$-subalgebra

$$\mathcal{A}_Q := \langle \bigcup_{(\mathbf{x}', Q') \sim (\mathbf{x}, Q)} \mathbf{x}' \rangle \subset \mathcal{F}.$$

Theorem (Fomin–Zelevinsky 2001)

The cluster algebra $\mathcal{A}_Q$ only depends on the mutation class of Q.

Example: cluster algebra



$$\mathcal{A}_Q = \left\langle x_1, x_2, \frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1x_2}, \frac{1+x_1}{x_2} \right\rangle \subset \mathbb{C}(x_1, x_2).$$

Structure Theorems

Theorem (Fomin–Zelevinsky 2001)

All cluster variables are Laurent polynomials in the cluster variables of the initial seed with integer coefficients. More precisely, they are contained in

$$\mathbb{Z}[x_1^{\pm}, \ldots, x_n^{\pm}, x_{n+1}, \ldots, x_{n+m}].$$

Positivity Conjecture (Fomin–Zelevinsky 2001)

All cluster variables are contained in $\mathbb{N}[x_1^{\pm}, \ldots, x_n^{\pm}, x_{n+1}, \ldots, x_{n+m}]$.

Theorem (Gross–Hacking–Keel–Kontsevich 2014)

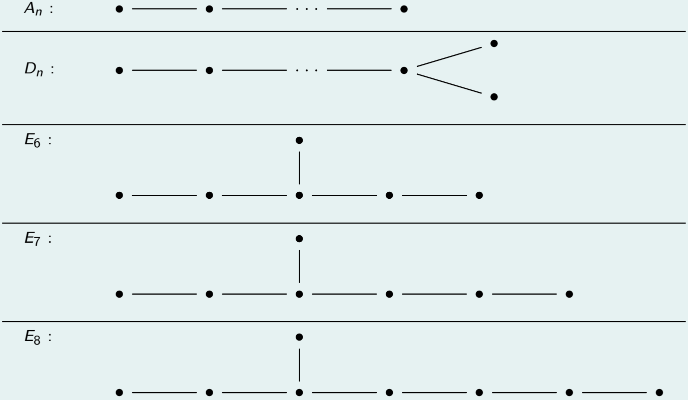
The positivity conjecture is true.

Gross–Hacking–Keel–Kontsevich view cluster algebras as rings of functions on certain log Calabi–Yau varieties (cluster varieties) and use tools from birational geometry and mirror symmetry.

Finite type classification

Theorem (Fomin–Zelevinsky 2003)

A cluster algebra $\mathcal{A}_Q$ is of finite type *(i.e. the set of its cluster variables is finite) if and only if (the mutable part of) Q is mutation equivalent to an orientation of a type* `ADE` *Dynkin diagram:*



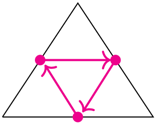
Grassmannian $\mathrm{Gr}_2(\mathbb{C}^n)$ and Ptolemy

Recall the $\binom{n}{2}$ Plücker coordinates p_{ij} for $\mathrm{Gr}_2(\mathbb{C}^n)$ and the correspondence:

$$\left\{\begin{matrix}\text{efficient TP}\\ \text{tests for } \mathrm{Gr}_2(\mathbb{C}^n)\end{matrix}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{matrix}\text{triangulations}\\ \text{of an } n\text{-gon}\end{matrix}\right\}.$$

We can pass from a triangulation T to a quiver Q_T as follows:

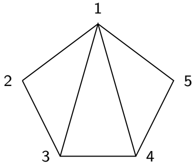
1. mutable vertices of $Q_T \leftrightarrow$ diagonals;
2. frozen vertices of $Q_T \leftrightarrow$ boundary edges;
3. add arrows inside every triangle:



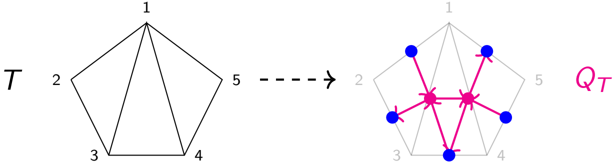
4. forget arrows between frozen vertices.

Example: $\mathrm{Gr}_2(\mathbb{C}^5)$

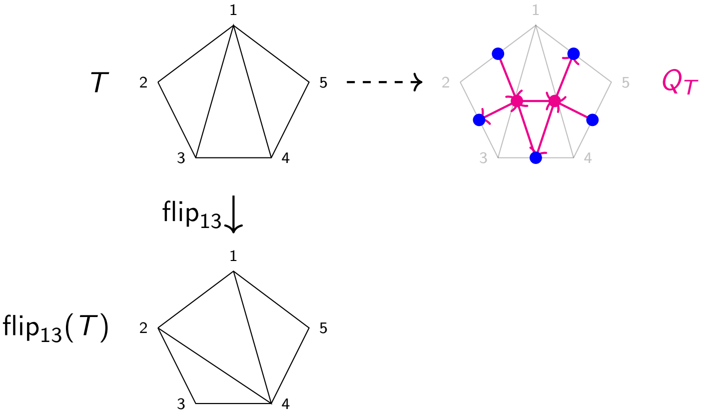
T



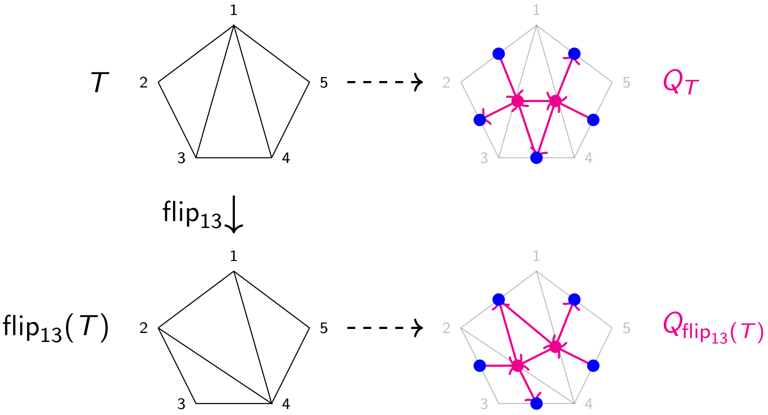
Example: $\mathrm{Gr}_2(\mathbb{C}^5)$



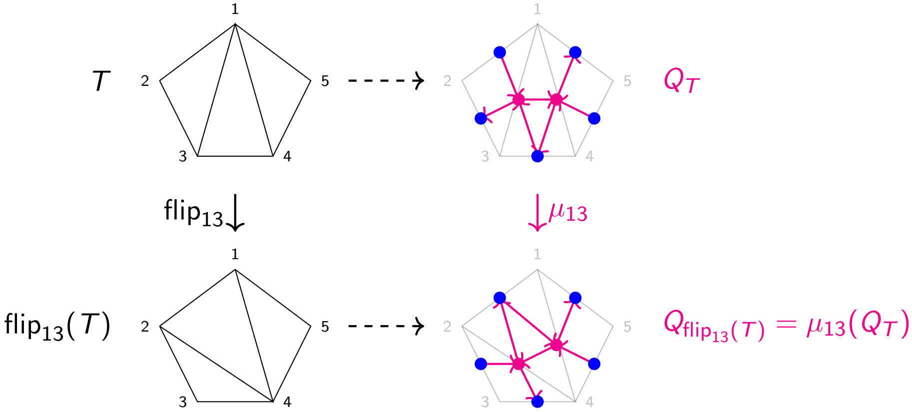
Example: $\mathrm{Gr}_2(\mathbb{C}^5)$



Example: $Gr_2(\mathbb{C}^5)$



Example: $\mathrm{Gr}_2(\mathbb{C}^5)$



Example: cluster algebra for $\mathrm{Gr}_2(\mathbb{C}^n)$

We have a bijection

$$\left\{ \begin{matrix} \text{triangulations} \\ \text{of an } n\text{-gon} \end{matrix}, \textit{flip} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{matrix} \text{seeds of the cluster} \\ \text{algebra } \mathcal{A}_{Q_T} \end{matrix}, \textit{mutation} \right\}.$$

In particular: $\{\text{seeds}\} \overset{1-1}{\longleftrightarrow} \{\text{efficient TP tests}\}$.

The corresponding cluster algebra is

$$\mathcal{A}_{Q_T} = \mathbb{C}[p_{ij} : 1 \leq i < j \leq n]/\text{Plücker relations}.$$

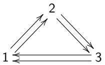
Remark

Similar results hold for $\mathrm{Gr}_k(\mathbb{C}^n)$, GL_n, SL_n, (partial) flag varieties, Schubert varieties, double Bruhat cells, ...

Remark

The cluster structure can be used to get (for example) Landau–Ginzburg models, toric degenerations and Newton–Okounkov bodies.

More cluster structures

1. Marked bordered surfaces:
   - fix a triangulation $\to$ coordinates for the Teichmüller space,
   - triangulation $\leftrightarrow$ quiver,
   - flip of triangulation $\leftrightarrow$ quiver mutation $\leftrightarrow$ Ptolemy relation.
2. 2-bridge knots and links:
   - 2-bridge link $\leftrightarrow$ continued fraction $\leftrightarrow$ snake graph $\leftrightarrow$ cluster variable of some cluster algebra,
   - Jones polynomial $=$ specialization of the Laurent polynomial of the cluster variable.
3. Markov equation: $x_1^2 + x_2^2 + x_3^2 = 3x_1x_2x_3$,
   - solutions are called Markov triples,
   - take $Q =$ 

 and $(x_1, x_2, x_3) = (1, 1, 1)$,
   - $(\mathbf{x}', Q) \sim (\mathbf{x}, Q)$ then $\mathbf{x}' = (x_1', x_2', x_3')$ is a Markov triple.
4. ...

References

FZ02 Sergey Fomin and Andrei Zelevinsky. Cluster algebras. I. Foundations. *J. Amer. Math. Soc.*, 15(2):497–529, 2002.

FZ03 Sergey Fomin and Andrei Zelevinsky. Cluster algebras. II. Finite type classification. *Invent. Math.* 154 (2003), no. 1, 63–121.

BFZ05 Arkady Berenstein, Sergey Fomin and Andrei Zelevinsky. Cluster algebras. III. Upper bounds and double Bruhat cells. Duke Math. J. 126 (2005), no. 1, 1–52.

FZ07 Sergey Fomin and Andrei Zelevinsky. Cluster algebras. IV. Coefficients. *Compos. Math.* 143, no. 1, 112–164 (2007)

FG07 Vladimir V. Fock and Alexander Goncharov. Dual Teichmüller and lamination spaces. Handbook of Teichmüller theory. Vol. I, 647–684, *IRMA Lect. Math. Theor. Phys.*, 11, Eur. Math. Soc., Zürich, (2007)

GHKK18 Mark Gross, Paul Hacking, Sean Keel, and Maxim Kontsevich. Canonical bases for cluster algebras. *J. Amer. Math. Soc.*, 31(2):497–608 (2018)

Pen87 Robert C. Penner. The decorated Teichmüller space of punctured surfaces. *Comm. Math. Phys.* Volume 113, Number 2 (1987), 299-339.

Sco06 Joshua S. Scott. Grassmannians and cluster algebras. *Proc. London Math. Soc.* (3) 92 (2006), no. 2, 345–380.

LS19 Kyungyong Lee and Ralf Schiffler. Cluster algebras and Jones polynomials. *Selecta Math. (N.S.)* 25 4 Paper No. 58, 41 (2019)