Gröbner degenerations of Grassmannains and cluster algebras

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Overview

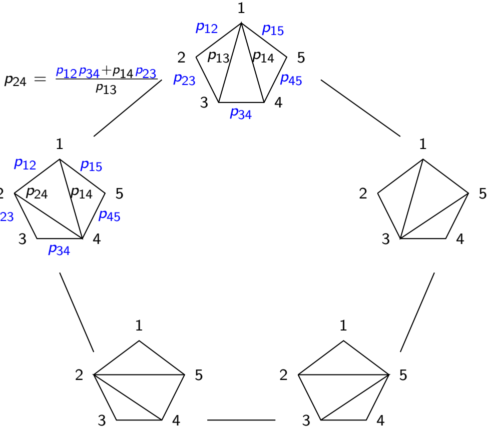
1. Cluster algebras
    1. Motivation: $\mathrm{Gr}_2(\mathbb{C}^n)$
    2. Quivers, seeds, mutation
    3. Cluster algebras: structure and classification results
    4. Grassmannian cluster algebra
    5. Principal and universal coefficients
2. Gröbner degenerations

Motivation: Grassmannian $\mathrm{Gr}_2(\mathbb{C}^5)$

The homogeneous coordinate ring of $\mathrm{Gr}_2(\mathbb{C}^5)$ with its Plücker embedding:

$$A_{2,5} := \mathbb{C}[p_{ij} : 1 \leq i < j \leq 5]/(p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk})_{1 \leq i<j<k<l \leq 5}$$

can be constructed recursively from triangulations of a 5-gon:



This is the prototype of a cluster algebra!

Quivers and mutation

A *quiver* Q is a directed graph, consisting of a finite set of vertices and arrows between them.

Technical assumption: Q does not have any loops or 2-cycles.

We split the vertex set into *mutable vertices* $\{1, \dots, n\}$ and *frozen vertices* $\{n+1, \dots, m\}$, e.g. $1 \rightrightarrows 2 \to 3$.

Definition (Quiver mutation)

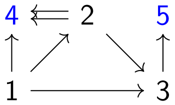
Given a quiver Q and a mutable vertex k, the *mutation in direction k* $\mu_k(Q)$ is a quiver obtained from Q in three steps:

1. for every path $i \to k \to j$ add an arrow $i \to j$;
2. invert every arrow incident to k;
3. remove a maximal set of 2-cycles.

Exercise: Quiver mutation is an *involution*: $\mu_k(\mu_k(Q)) = Q$.

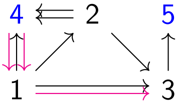
Example: quiver mutation

Q :

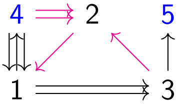


$\mu_2(Q)$:

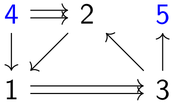
Step 1:



Step 2:

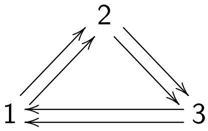


Step 3:



: $\mu_2(Q)$

Exercise:



is invariant under mutation/isomorphism.

Seeds and mutation

A *seed* s is a pair $(\mathbb{x}, Q)$, where $\mathbb{x} = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+m})$ is a collection of variables called a *cluster* and Q a quiver with n mutable and m frozen vertices.

Definition (Seed mutation)

Given a seed $s = (\mathbb{x}, Q)$ and a mutable vertex k of Q, the *mutation in direction k* $\mu_k(s)$ is the pair $(\mu_k(\mathbb{x}), \mu_k(Q))$, where $\mu_k(\mathbb{x}) = \mathbb{x} \setminus \{x_k\} \cup \{x_k'\}$ and

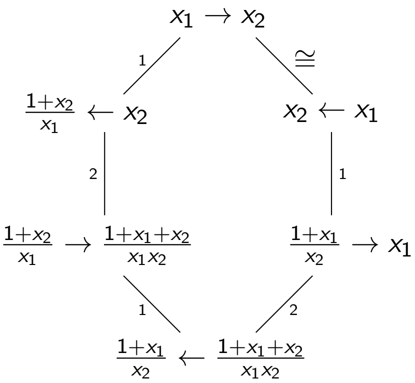
$$x_k' := \frac{\prod_{i \to k \in Q} x_i + \prod_{k \to j \in Q} x_j}{x_k}. \tag{0.1}$$

The equation (0.1) is called en *exchange relation*.

Exercise: Seed mutation is an *involution*: $\mu_k(\mu_k(s)) = s$.

Notation: $s = (\{x_1, x_2\}, 1 \to 2) = (x_1 \to x_2)$.

Example: seed mutation



cluster magic!

$$\left(1+\frac{1+x_1+x_2}{x_1x_2}\right) \div \frac{1+x_2}{x_1} = \frac{x_1x_2+1+x_1+x_2}{x_2(1+x_2)} = \frac{1+x_1}{x_2}$$

Cluster algebra

Let $\mathcal{F} = \mathbb{C}(x_1, \ldots, x_{n+m})$ be the field of rational functions in the variables $x_1, \ldots, x_{n+m}$.

If $s = (\mathbb{x}, Q)$ is a seed with $\mathbb{x} = (x_1, \ldots, x_{n+m})$ and $s' = (\mathbb{x}', Q')$ a seed obtained from s by a sequence of mutations, then the cluster $\mathbb{x}' = (x'_1, \ldots, x'_n, x'_{n+1}, \ldots, x'_{n+m})$ satisfies

$$\mathbb{C}(x'_1, \ldots, x'_n, x'_{n+1}, \ldots, x'_{n+m}) = \mathcal{F}.$$

Definition

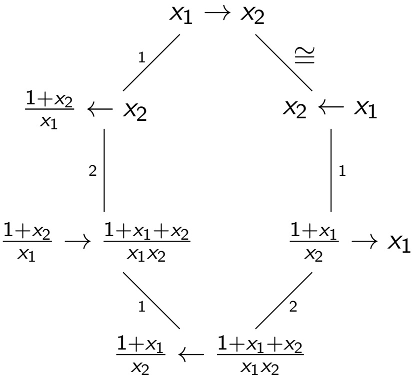
The *cluster algebra* defined by the initial seed s is the $\mathbb{C}$-subalgebra

$$A_s := \langle \bigcup_{s'=(\mathbb{x}', Q') \sim s} \mathbb{x}' \rangle \subset \mathcal{F}.$$

Theorem (Fomin–Zelevinsky 2001)

The cluster algebra A_s only depends on the mutation class of Q.

Example: cluster algebra



$$A_{x_1\to x_2} = \left\langle x_1, x_2, \frac{1+x_2}{x_1}, \frac{1+x_1+x_2}{x_1x_2}, \frac{1+x_1}{x_2} \right\rangle \subset \mathbb{C}(x_1,x_2).$$

Structure Theorems

Theorem (Fomin–Zelevinsky 2001)

All cluster variables are Laurent polynomials in the cluster variables of the initial seed with integer coefficients. More precisely, they are contained in

$$\mathbb{Z}[x_1^{\pm}, \dots, x_n^{\pm}, x_{n+1}, \dots, x_{n+m}].$$

Positivity Conjecture (Fomin–Zelevinsky 2001)

All cluster variables are contained in $\mathbb{N}[x_1^{\pm}, \dots, x_n^{\pm}, x_{n+1}, \dots, x_{n+m}]$.

Theorem (Gross–Hacking–Keel–Kontsevich 2014)

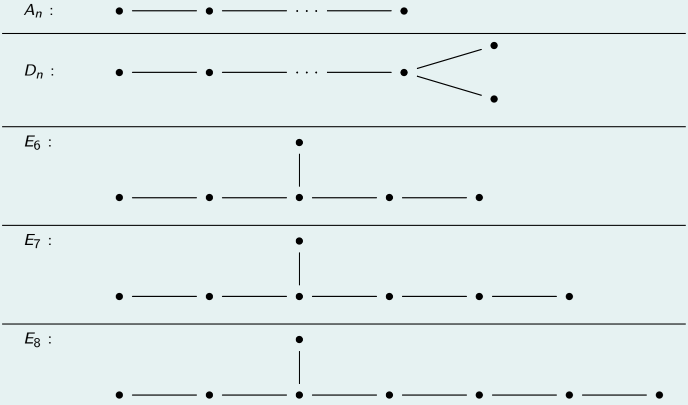
The positivity conjecture is true.

Gross–Hacking–Keel–Kontsevich view cluster algebras as rings of functions on certain log Calabi–Yau varieties (cluster varieties) and use tools from birational geometry and mirror symmetry.

Finite type classification

Theorem (Fomin–Zelevinsky 2003)

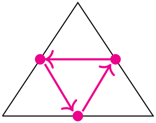
A cluster algebra $A_{(\mathbb{x},Q)}$ is of finite type (i.e. the set of its cluster variables is finite) if and only if (the mutable part of) Q is mutation equivalent to an orientation of a type ADE *Dynkin diagram:*



Grassmannian $\mathrm{Gr}_2(\mathbb{C}^n)$

Recall the $\binom{n}{2}$ Plücker coordinates p_{ij} for $\mathrm{Gr}_2(\mathbb{C}^n)$. We can pass from a triangulation T to a quiver Q_T as follows:

1. mutable vertices of $Q_T \leftrightarrow$ diagonals;
2. frozen vertices of $Q_T \leftrightarrow$ boundary edges;
3. add arrows inside every triangle:



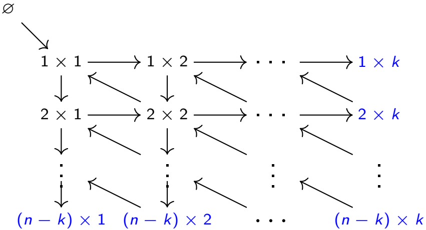
4. forget arrows between frozen vertices.
5. We have a bijection

$$\left\{ \begin{matrix} \text{triangulations } T \\ \text{of an } n\text{-gon} \end{matrix}, \; \textit{flip} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{matrix} \text{seeds of the cluster} \\ \text{algebra } A_T \end{matrix}, \; \textit{mutation} \right\}.$$

Exercise: Verify that the *flip* translates to the *mutation*.

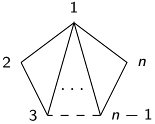
Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$

For a general Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$ we define a seed $s = (Q, \mathbb{x})$ with $\mathbb{x} = (x_{i\times j})_{i,j}$ where $x_{i\times j} := p_{[1,k-j]\cup[k-j+i+1,k+i]}$ and quiver Q:



Exercise: for $k = 2$

Q corresponds to



Mutation at 4-valent vertices $\leftrightarrow$ Plücker relations, e.g.

$$\mu_{1\times 1}(x_{1\times 1}) = \frac{x_{1\times 2}x_{2\times 1} + x_{\varnothing}x_{2\times 2}}{x_{1\times 1}}$$

$$= \frac{p_{[1,k-2]\cup[k,k+1]}p_{[1,k-2]\cup[k+1,k+2]} + p_{[1,k-1]\cup\{k+1\}}p_{[1,k]}}{p_{[1,k-1]\cup\{k+1\}}} = p_{[1,k-2]\cup\{k,k+2\}}.$$

Theorem (Scott 2006)

The cluster algebra A_s is isomorphic to the homogeneous coordinate ring of the Grassmannian with respect to its Plücker embedding:

$$A_s \cong A_{k,n} = \mathbb{C}\left[p_J : J = \{j_1, \ldots, j_k\} \subset [n]\right] / I_{k,n}$$

- for $k \leq 2$ Plücker coordinates = cluster variables.
- for $k \geq 3$ Plücker coordinates $\subsetneq$ cluster variables.
- if $k = 2$ ($\mathtt{A}_{n-3}$) or $k = 3, n \in \{6, 7, 8\}$ ($\mathtt{D}_4, \mathtt{E}_6, \mathtt{E}_8$) finitely many seeds and in all other cases there are infinitely many seeds.

Remark

Similar results hold for double Bruhat cells [Berenstein–Fomin–Zelevinksy 2005], (partial) flag varieties [Geiss–Leclerc–Schröer 2008], (open) Richardson varieties [Leclerc 2016] and Schubert varieties [Sherman-Bennett–Serhiyenko–Williams 2020].

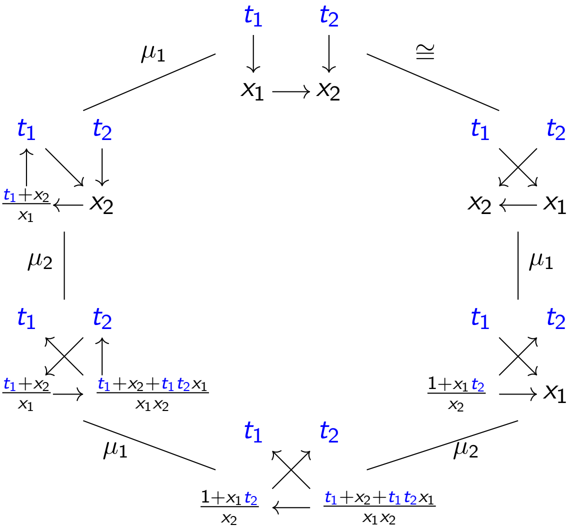
Principal coefficients

Definition (Fomin–Zelevinsky 2005)

Given a cluster algebra A with initial seed $s = (Q, \mathbb{x})$ we define the corresponding *cluster algebra with principal coefficients at s*, denoted A_s^{prin}, by the initial seed $\hat{s} := (\hat{Q}, \hat{\mathbb{x}})$, where

- $\hat{Q}$ has vertices $v_1, \dots, v_n$ (as Q) and frozen vertices $v_1', \dots, v_n'$ and contains Q as a full subquiver, additionally there are arrows $v_i' \to v_i$ for all i;
- $\hat{\mathbb{x}} = (\mathbb{x}, \mathbb{t})$ where $\mathbb{t} = (t_1, \dots, t_n)$.

Example: principal coefficients



$$\Rightarrow \quad A^{\text{prin}}_{x_1\to x_2} = \left\langle x_1, x_2, \frac{t_1+x_2}{x_1}, \frac{t_1+t_1t_2x_1+x_2}{x_1x_2}, \frac{1+x_1t_2}{x_2} \right\rangle$$

Observations:

1. A_s and A_s^{prin} have the same *seed pattern*;
2. $A_s^{\text{prin}} \subset \mathbb{Z}[t_1, t_2][x_1^{\pm1}, x_2^{\pm1}]$;
3. for $(1,0) := \deg(x_1) := -\deg(t_2)$ and $(0,1) := \deg(x_2) := \deg(t_1)$ all cluster variables are homogeneous, so A_s^{prin} is $\mathbb{Z}^2$-graded:

$$\deg\left(\tfrac{t_1+x_2}{x_1}\right) = (-1,1), \quad \deg\left(\tfrac{t_1+t_1t_2x_1+x_2}{x_1x_2}\right) = (-1,0), \quad \deg\left(\tfrac{1+x_1t_2}{x_2}\right) = (0,-1).$$

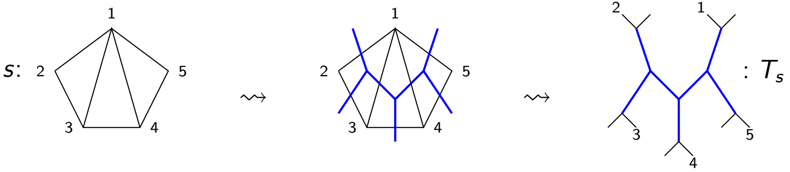
Theorem (Fomin–Zelevinsky 2005)

Given a cluster algebra A_s with initial seed $s = (Q, \mathbb{x})$ and the corresponding cluster algebra with principal coefficients A_s^{prin} at s, then

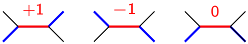
1. *A_s^{prin} has the same seed pattern as A_s,*
2. *$A_s^{prin} \subset \mathbb{Z}[t_1, \dots, t_n][x_1^{\pm1}, \dots, x_n^{\pm1}]$,*
3. *A_s^{prin} is graded with grading induced by $\deg(x_i) = e_i \in \mathbb{Z}^n$ and $\deg(t_i) := -\sum_{j=1}^n \#\{i \to j\} e_j$; every cluster variable x is a homogeneous elements and its degree called g-vector.*

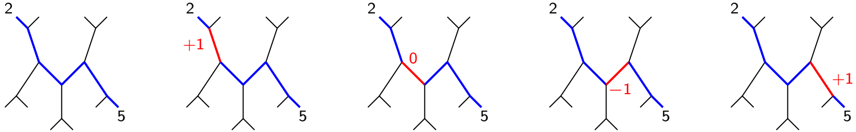
Example: g-vectors for $\mathrm{Gr}_2(\mathbb{C}^n)$

Consider the triangulation of a seed s of $A_{2,n}$:



To compute $g_s(p_{ij}) = \sum_{p_{ab}\in s} c_{ab}e_{ab}$ take the path ρ_{ij} from i to j in the tree T_s. The coefficients can be read from ρ_{ij}:





$\Rightarrow g_s(p_{25}) = e_{12} - e_{14} + e_{45}.$

g-vector valuation

Proposition (GHKK, Fujita–Oya, B–Cheung–Magee–Nájera Chávez)

Let s' be an arbitrary seed of $A_s = A_{k,n}$ and x denote any cluster variable, then

$$g_{s'} : A_{k,n} \setminus \{0\} \to \mathbb{Z}^{k(n-k)+1} \quad \text{with} \quad x \mapsto g_s(x)$$

is a (full-rank homogeneous) valuation with finitely generated value semigroup. As such it defines a toric degeneration of $Spec(A_{k,n})$, the affine cone of $\mathrm{Gr}_k(\mathbb{C}^n)$. Moreover, g_s has a $\mathbb{C}[t_{i\times j} : i, j]$-basis (adapted to g_s) called the ϑ-basis that is independent of s.

Remark: The Proposition holds more generally for any cluster algebra that satisfies the *full Fock–Goncharov conjecture*.

Newton–Okounkov bodies for Grassmannains

Proposition

For every seed s of $A_{2,n}$ the value semigroup $im(g_s)$ is generated by the g-vectors of Plücker coordinates. Moreover, the Newton–Okounkov body of g_s is the convex hull

$$\Delta(A_{2,n}, g_s) = conv\{g_s(p_{ij}) : 1 \leq i < j \leq n\}.$$

Remark (B.–Cheung–Magee–Nájera Chávez)

For arbitrary $\mathrm{Gr}_k(\mathbb{C}^n)$ Rietsch–Williams define a valuation $v_s : A_{k,n} \to \mathbb{Z}^{k(n-k)}$ for every plabic graph s (or more generally for every seed s of $A_{k,n}$). We can show that

$$\Delta(A_{k,n}, g_s) \cong \Delta(A_{k,n}, v_s).$$

Moreover, the Newton–Okounkov cone of g_s is the polyhedral cone obtained from tropicalizing GHKK's superpotential.

Universal coefficients

Definition (Fomin–Zelevinsky 2005, Reading 2014)

Given a cluster algebra A_s of finite type with initial seed $s = (Q, \mathbb{x})$. Let $\{x_1, \dots, x_N\}$ be the set of *all* cluster variables. Then the corresponding *cluster algebra with universal coefficients* A_s^{univ} has initial seed $s^{\text{univ}} = (Q^{\text{univ}}, \mathbb{x}^{\text{univ}})$, where

- Q^{univ} has vertices $v_1, \dots, v_n$ (as Q) and frozen vertices $v'_1, \dots, v'_N$ and contains Q as a full subquiver, additionally there are arrows defined as follows: let $g_{s^{op}}(x_i) \in \mathbb{Z}^n$ be the g-vector of x_i for $1 \leq i \leq N$ with respect to the *opposed* quiver Q^{op}, then[a]

$$\#\{v'_i \to v_j\} - \#\{v_j \to v'_i\} := g_{s^{op}}(x_i)_j.$$

- $\mathbb{x}^{\text{univ}} = (\mathbb{x}, \mathbb{t}^{\text{univ}})$ where $\mathbb{t}^{\text{univ}} = (t_1, \dots, t_N)$.

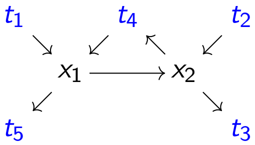
[a] Remember that our quivers are not allowed to have 2-cycles.

$$A_{x_1 \to x_2} = \left\langle x_1, x_2, x_3 := \frac{1+x_2}{x_1}, x_4 := \frac{1+x_1+x_2}{x_1 x_2}, x_5 := \frac{1+x_1}{x_2} \right\rangle$$

$$g_i := g_{x_1 \leftarrow x_2}(x_i)$$

Exercise:

$$\begin{aligned} g_1 &= (\ \ 1,\ \ 0) \\ g_2 &= (\ \ 0,\ \ 1) \\ g_3 &= (\ \ 0,-1) \\ g_4 &= (\ \ 1,-1) \\ g_5 &= (-1,\ \ 0) \end{aligned}$$



Theorem (Fomin–Zelevinsky 2005)

Let A be a cluster algebra of finite type and A^{univ} the cluster algebra with universal coefficients, then

1. $A^{univ} \subset \mathbb{C}[t_1, \ldots, t_N][x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$;
2. A^{univ} *is independent of the initial seed;*
3. *for every seed s there exists a unique map $A^{univ} \to A_s^{prin}$ sending cluster variables to cluster variables.*

Overview

1. Cluster algebras
2. Gröbner degenerations
   1. Initial ideals, Gröbner fan and degenerations
   2. A family of Gröbner degenerations
   3. Toric degenerations, from valuation to tropicalization
   4. g-vector valuation of finite type
   5. Universal coefficients and a Gröbner cone

Initial ideals

Let $f = \sum c_\alpha \mathbb{x}^\alpha \in \mathbb{C}[x_1, \dots, x_n]$ with $c_\alpha \in \mathbb{C}$, $\alpha \in \mathbb{Z}^n_{\geq 0}$ and $\mathbb{x}^\alpha := x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n}$.

For $w \in \mathbb{R}^n$ we define its *initial form with respect to w* as

$$\mathrm{in}_w(f) := \sum_{\beta : w \cdot \beta = \min_{c\alpha \neq 0}\{w \cdot \alpha\}} c_\beta \mathbb{x}^\beta.$$

For $J \subset \mathbb{C}[x_1, \dots, x_n]$ an ideal we define its *initial ideal with respect to w* as $\mathrm{in}_w(J) := (\mathrm{in}_w(f) : f \in J)$.

Example

For $f = x_1^2 x_2^2 + x_1^4 + x_2 x_3^3 \in \mathbb{C}[x_1, x_2, x_3]$ and $w = (1, 0, 0)$ we compute

$$\mathrm{in}_w(f) = x_2 x_3^3.$$

Gröbner fan and Gröbner degenerations

Definition/Proposition (Mora–Robbiano 1988)

For a homogeneous ideal $J \subset \mathbb{C}[x_1, \ldots, x_n]$ its *Gröbner fan* $GF(J)$ is $\mathbb{R}^n$ with fan structure defined by

$$v, w \in C^\circ \quad \Leftrightarrow \quad \mathrm{in}_v(J) = \mathrm{in}_w(J).$$

Notation: $\mathrm{in}_C(J) := \mathrm{in}_w(J)$ for any $w \in C^\circ$.

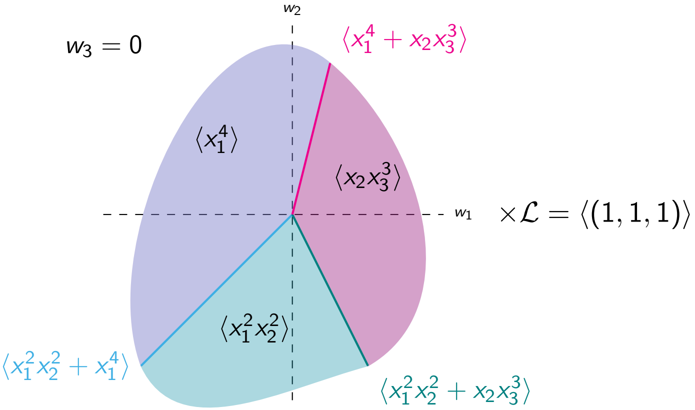
Every open cone $C^\circ \in GF(J)$ defines a *Gröbner degeneration*

$$\pi : \mathcal{V} \to \mathbb{A}^1$$

with $\pi^{-1}(t) \cong V(J)$ for $t \neq 0$ and $\pi^{-1}(0) = V(\mathrm{in}_C(J))$.

Example

Take $I = (x_1^2x_2^2 + x_1^4 + x_2x_3^3) \subset \mathbb{C}[x_1, x_2, x_3]$. Then $GF(I)$ is $\mathbb{R}^3$ with the fan structure:



Standard monomial basis

Let $A := \mathbb{C}[x_1, \dots, x_n]/J$ and $A_\tau := \mathbb{C}[x_1, \dots, x_n]/\mathrm{in}_\tau(J)$ for $\tau \in GF(J)$.

Fix a maximal cone $C \in GF(J)$, then the ideal $\mathrm{in}_C(J)$ is generated by monomials. For every face $\tau \subseteq C$ we define

$$\mathbb{B}_{C,\tau} := \{\bar{\mathbb{x}}^\alpha \in A_\tau \mid \mathbb{x}^\alpha \notin \mathrm{in}_C(J)\}.$$

Then $\mathbb{B}_{C,\tau}$ is a vector space basis for A_τ called *standard monomial basis*. In particular, $\mathbb{B}_C := \mathbb{B}_{C,\{0\}}$ is a vector space basis for $A = A_{\{0\}}$.

$\rightsquigarrow$ All degenerations $\{A_\tau : \tau \subseteq C\}$ have a basis indexed by $\mathbb{x}^\alpha \notin \mathrm{in}_C(J)$.

Family of ideals

Let $C \in GF(J)$ be a maximal cone and choose $r_1, \dots, r_m$ representatives of primitive ray generators of $\overline{C} \in \mathrm{GF}(J)/\mathcal{L}$. Let r be the matrix with rows $r_1, \dots, r_m$. Define for $f = \sum c_\alpha \mathbb{x}^\alpha \in J \subset \mathbb{C}[x_1, \dots, x_n]$

$$\mu(f) := (\min_{c_\alpha \neq 0}\{r_1 \cdot \alpha\}, \dots, \min_{c_\alpha \neq 0}\{r_m \cdot \alpha\}) \in \mathbb{Z}^m.$$

In $\mathbb{C}[t_1, \dots, t_m][x_1, \dots, x_n]$ we define the *lift* of f

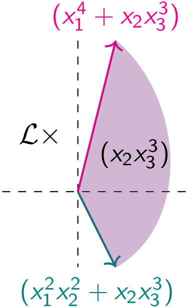
$$\tilde{f} := f(\mathbb{t}^{r \cdot e_1} x_1, \dots, \mathbb{t}^{r \cdot e_n} x_n)\mathbb{t}^{-\mu(f)} = \sum c_\alpha \mathbb{x}^\alpha \mathbb{t}^{r \cdot \alpha - \mu(f)}.$$

Definition/Proposition

The *lifted ideal* $\tilde{J} := (\tilde{f} : f \in J) \subset \mathbb{C}[t_1, \dots, t_m][x_1, \dots, x_n]$ is generated by $\{\tilde{g} : g \in \mathcal{G}\}$, where $\mathcal{G}$ is a *Gröbner basis* for J and C.

Example

Take $f = x_1^2 x_2^2 + x_1^4 + x_2 x_3^3 \in \mathbb{C}[x_1, x_2, x_3]$ and consider in $GF((f))$ the maximal cone C spanned by the rows of $r := \begin{pmatrix} 1 & 4 & 0 \\ 1 & -2 & 0 \end{pmatrix}$ and $\mathcal{L}$. We compute



$$(\mathrm{t}_1, t_2) = f(t_1 t_2 x_1, t_1^4 t_2^{-2} x_2, x_3) t_1^{-4} t_2^2$$
$$= \mathrm{t}_1^6 x_1^2 x_2^2 + t_2^6 x_1^4 + x_2 x_3^3$$

- $\tilde{f}(0, 0) = x_2 x_3^3 = \mathrm{in}_C(f)$,
- $\tilde{f}(0, 1) = x_1^4 + x_2 x_3^3 = \mathrm{in}_{r_1}(f)$,
- $\tilde{f}(1, 0) = x_1^2 x_2^2 + x_2 x_3^3 = \mathrm{in}_{r_2}(f)$,
- $\tilde{f}(1, 1) = f$.

A family of Gröbner degenerations

Let $\tilde{A} := \mathbb{C}[t_1, \ldots, t_m][x_1, \ldots, x_n]/\tilde{J}$, $A_\tau = \mathbb{C}[x_1, \ldots, x_n]/\mathrm{in}_\tau(J)$.

Theorem (B.–Mohammadi–Nájera Chávez)

$\tilde{A}$ is a free $\mathbb{C}[t_1, \ldots, t_m]$-module with standard monomial basis $\mathbb{B}_C$ and so the morphism

$$\pi : Spec(\tilde{A}) \to \mathbb{A}^m$$

is free. In particular, π defines a flat family with generic fiber $Spec(A)$ and for every face $\tau \subseteq C$ there exists $a_\tau \in \mathbb{A}^m$ and a special fiber $\pi^{-1}(a_\tau) \cong Spec(A_\tau)$.

Example

$$\tilde{A} = \mathbb{C}[t_1, t_2][x_1, x_2, x_3]/(t_1^6 x_1^2 x_2^2 + t_2^6 x_1^4 + x_2 x_3^3).$$

Toric degenerations

$GF(J)$ contains a subfan of dimension $\dim_{\mathrm{Krull}} A$ called the *tropicalization of J*

$$\mathrm{Trop}(J) := \{w \in \mathbb{R}^n : \mathrm{in}_w(J) \not\ni \text{monomials}\}.$$

Corollary (B.–Mohammadi–Nájera Chávez)

Consider the fan $\Sigma := C \cap Trop(J)$. If there exists $\tau \in \Sigma$ with $in_\tau(J)$ binomial and prime, then the family

$$\pi : Spec(\tilde{A}) \to \mathbb{A}^m$$

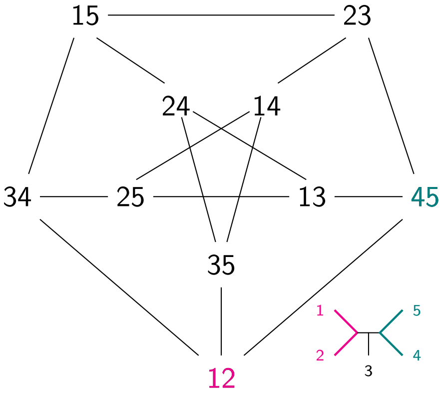
contains toric fibers isomorphic to $Spec(A_\tau)$ (affine toric scheme).

Remark

The degenerations $\mathrm{Spec}(A_\tau)$ can be obtained from valuations and Newton–Okounkov bodies [Kaveh–Manon 2019].

Example: tropical Grassmannian $Gr_2(\mathbb{C}^5)$

For $n = 5$, the tropical Grassmannian $\text{Trop}(I_{2,5})$ is a 7-dimensional fan in $\mathbb{R}^{10}$ with a 5-dimensional linear subspace $\mathcal{L}$.
As a complex it coincides with the Peterson graph:



The 15 maximal cones are in bijection with trivalent trees with 5 leaves.

From valuation to tropicalization

Theorem (B. 2020, Kaveh–Manon 2019)

Let A be an algebra and domain of Krull dimension d and $v : A \setminus \{0\} \to \mathbb{Z}^d$ a full-rank valuation with finitely generated value semigroup:[a]

$$im(v) = \langle v(b_1), \ldots, v(b_N) \rangle^b.$$

Consider $\pi : \mathbb{C}[x_1, \ldots, x_N] \to A$ with $\pi(x_i) = b_i$ and set $J := \ker(\pi)$. Then $A \cong \mathbb{C}[x_1, \ldots, x_N]/J$ and there exists a maximal cone $\tau \subset Trop(J)$:

$$A_\tau \cong \mathbb{C}[im(v)].$$

[a] [Anderson] So the semigroup algebra $\mathbb{C}[im(v)]$ is toric and $Spec(\mathbb{C}[im(v)])$ is a toric degeneration of $Spec(A)$.

[b] In this case $\{b_1, \ldots, b_N\}$ is called a Khovanskii basis for A and v.

We call the ideal J and the presentation $A \cong \mathbb{C}[x_1, \ldots, x_N]/J$ *adapted* to the valuation v.

From g-vector valuation to tropicalization

Recall the g-vector valuation on the Grassmannian cluster algebra $A_{k,n}$. For every seed s we have

$$g_s : A_{k,n} \setminus \{0\} \to \mathbb{Z}^{k(n-k)+1}$$

[GHKK]: The value semigroup $im(g_s)$ is finitely generated for all s.

Questions:

1. How do we find an ideal adapted to a g-vector valuation g_s?
2. Does there exist an ideal adapted to g_s for all s simultaneously?
3. If so, is there a relation to the cluster algebra with universal coefficients?

g-vectors for finite type cluster algebras

Theorem (Fomin–Zelevinsky 2005, Cerulli Irelli–Keller–Labardini–Plamondon 2013, GHKK 2018)

Let A be a cluster algebra of finite type and s any seed. Then

1. *the cluster monomials*[a] *are a $\mathbb{C}$-basis for A, A_s^{prin} and A^{univ};*
2. *the g-vectors form a complete simplicial fan, called the g-fan, whose maximal cones correspond to seeds.*

[a] Monomials in cluster variables of the same seed.

For $\{x_1, \ldots, x_N\}$ cluster variables of A set

$$\pi : \mathbb{C}[y_1, \ldots, y_N] \to A, \quad \pi(y_i) = x_i.$$

Then $J = \ker(\pi)$ is adapted to g_s for all seeds s.
$\Rightarrow$ this answer questions 1 and 2 for finite type cluster algebras.

Finite type Grassmannains $Gr_3(\mathbb{C}^6)$ and $Gr_2(\mathbb{C}^n)$

$Gr_2(\mathbb{C}^n)$: cluster variables = Plücker coordinates $\Rightarrow J = I_{2,n}$ is generated by all exchange relations (= Plücker relations).

$Gr_3(\mathbb{C}^6)$: cluster variables $= \{\bar{p}_{123}, \dots, \bar{p}_{456}\} \cup \{\bar{X}, \bar{Y}\}$

$$\pi : \mathbb{C}[p_{123}, \dots, p_{456}, X, Y] \to A_{3,6} \quad \pi(p_{ijk}) = \bar{p}_{ijk}, \pi(X) = \bar{X}, \pi(Y) = \bar{Y},$$

so $J = \ker(\pi)$ is generated by all exchange relations and one 4-term Plücker relation.

Remark

The Plücker ideal $I_{3,6}$ for $Gr_3(\mathbb{C}^6)$ can be obtained from J by eliminating X, Y. However, $I_{3,6}$ is *not* adapted to g_s for all seeds s.

Theorem (B.–Mohammadi–Nájera Chávez 2020)

Let A be $A_{2,n}$ or $A_{3,6}$ and J the ideal adapted to the g-vector valuation g_s for all seeds s. Then there exists a unique maximal simplicial cone $C \subset GF(J)$ such that

1. *the fan $\Sigma = C \cap Trop(J)$ is combinatorially isomorphic to the g-fan:*

$$\{\text{rays of } C\} \leftrightarrow \{\text{cluster variables of } A\}$$
$$\left\{\begin{matrix}\text{maximal cones}\\ \tau_s \in C \cap Trop(J)\end{matrix}\right\} \leftrightarrow \{\text{seeds } s \text{ of } A\}$$

   and $\mathbb{C}[im(g_s)] \cong \mathbb{C}[x_1, \ldots, x_N]/in_{\tau_s}(J)$.

2. *we have a canonical isomorphism $\tilde{A} \cong A^{univ}$ identifying universal coefficients with rays of C;*
3. *the standard monomial basis $\mathbb{B}_C$ for A (and $\tilde{A}$) coincides with the basis of cluster monomials for A (and A^{univ});*
4. *the Stanley–Reisner ideal of the g-fan is the initial ideal $in_C(J)$.*

Stanley–Reisner ideals and complexes

Let Δ be a simplicial complex with vertex set $V = \{x_1, \dots, x_n\}$. The *Stanley–Reisner ideal* of Δ is

$$I_\Delta := \langle x_{i_1} \cdots x_{i_s} : \{x_{i_1}, \dots, x_{i_s}\} \notin \Delta \rangle \subset \mathbb{C}[x_1, \dots, x_n].$$

Let $A_\Delta := \mathbb{C}[x_1, \dots, x_n]/I_\Delta$, the *Stanley–Reisner scheme* is $Proj(A_\Delta)$. Reversely, to every square-free monomial ideal one can associate its *Stanley–Reisner complex*, whose non-faces are defined by the monomials in the ideal.

[Ilten]: the type $\mathtt{D}_n$ associahedron is *unobstructed*, hence the corresponding Stanley–Reisner scheme is a smooth point in its Hilbert scheme.

Corollary

The Grassmannian $Gr_3(\mathbb{C}^6)$, a cone over $\mathbb{P}(\mathtt{D}_4)$ (namely $Proj(A_C)$) and the toric schemes $Proj(A_{\tau_s})$ for all seeds s all lie on the same component of the Hilbert scheme.

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