Gröbner theory of Grassmannian cluster algebras

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Interdisciplinary applications of cluster algebras

Motivation

Idea: Want to relate a cluster algebra A with the Gröbner theory of an ideal I s.t. $A = k[x_1, \ldots, x_m]/I$.

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Theorem (Fomin–Williams–Zelevinsky 20)

Let a be a finite type cluster algebra with X_A the set of all cluster variables. Let $I_A \subset k[X_A]$ be the saturation of the ideal generated by all exchange relations. Then

$$A \cong k[X_A]/I_A.$$

Grassmannian cluster algebra

Let $A_{k,n}$ be the homogeneous coordinate ring of $\mathrm{Gr}(k,n)$ under its Plücker embedding

$$\mathrm{Gr(k,n)} \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}, \quad M \mapsto [\det(M_I)]_{I \in \binom{[n]}{k}}$$

[Scott 06] $A_{k,n}$ is a cluster algebra and of finite type if and only if $(k,n) \in \{(2,n),(3,6),(3,7),(3,8)\}$.

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Example

1. For $A_{2,n}$ all cluster variables are Plücker coordinates p_{ij} and the ideal $I_{A_{2,n}} = I_{2,n}$ is the Plücker ideal generated by

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2. For $A_{3,6}$ all Plücker coordinates p_{ijk} are cluster variables and there are two more: x, y that are quadratic binomials in p_{ijk}. So $I_{3,6}$ contains the Plücker ideal.

A minimal generating set of $I_{3,6} \subset \mathbb{C}[p_{123}, \ldots, p_{456}, x, y]$:

$$
\begin{array}{ll}
p_{145}p_{236} - p_{123}p_{456} - X, & p_{124}p_{356} - p_{123}p_{456} - Y, \\
p_{136}p_{245} - p_{126}p_{345} - X, & p_{125}p_{346} - p_{126}p_{345} - Y, \\
p_{146}p_{235} - p_{156}p_{234} - X, & p_{134}p_{256} - p_{156}p_{234} - Y, \\
p_{246}p_{356} - p_{346}p_{256} - p_{236}p_{456}, & p_{245}p_{356} - p_{345}p_{256} - p_{235}p_{456}, \\
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\end{array}
$$

$$f = p_{135}p_{246} - p_{156}p_{234} - Y - p_{123}p_{456} - X - p_{126}p_{345}.$$

Gröbner fan and (totally positive) tropicalization

Definition (Mora–Robbiano 88)

For a homogeneous ideal $J \subset \mathbb{C}[x_1, \dots, x_n]$ its *Gröbner fan* $GF(J)$ is $\mathbb{R}^n$ with fan structure defined by

$$v, w \in C^\circ \quad \Leftrightarrow \quad \mathrm{in}_v(J) = \mathrm{in}_w(J).$$

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There is a distinguished subfan of $GF(J)$ called the *tropicalization* of J:

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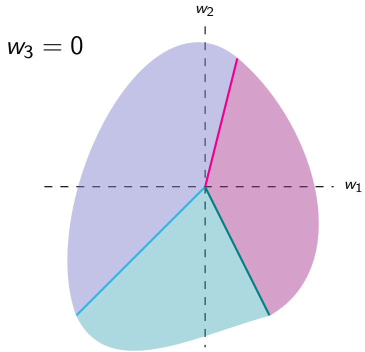
If $J \subset \mathbb{R}[x_1, \ldots, x_n]$ then the *totally positive part* $\mathrm{Trop}^+(J) \subset \mathrm{Trop}(J)$ is the closed subfan:

$$\mathrm{Trop}^+(J) := \{w \in Trop(J) : \mathrm{in}_w(J) \cap \mathbb{R}_+[x_1, \ldots, x_n] = \varnothing\}.$$

Example

Take $J = (x_1^2x_2^2 + x_1^4 - x_2x_3^3) \subset \mathbb{C}[x_1, x_2, x_3]$. Then $GF(J)$ is $\mathbb{R}^3$ with the fan structure and Trop(J) its one-skeleton, Trop$^+(J)$ is spanned by r_1, r_2:



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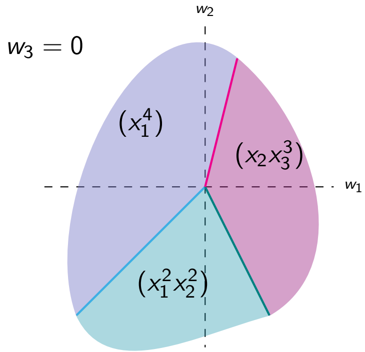

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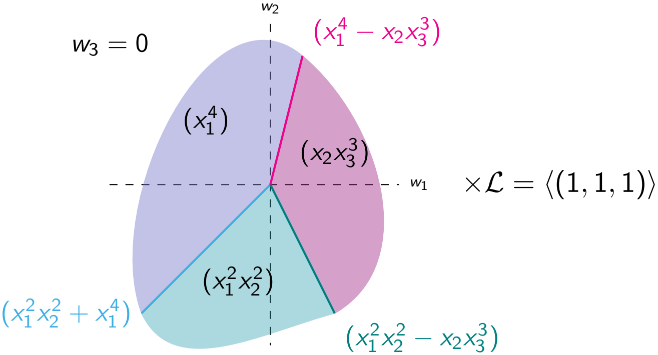

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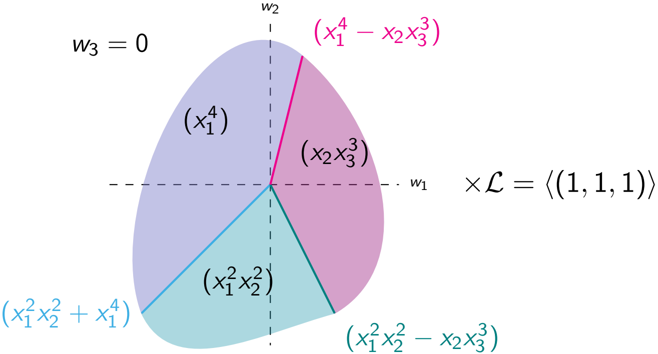

Note: $\mathbb{B}_{\langle r_1, r_2 \rangle} = \{\bar{\mathbf{x}}^a : x_2x_3^3 \nmid \mathbf{x}^a\}$ gives standard monomial basis for A.

Family of ideals

Let $C \in GF(J)$ be a maximal cone and choose $r_1, \dots, r_m$ representatives of primitive ray generators of $\overline{C} \in \mathrm{GF}(J)/\mathcal{L}$. Let $\mathbf{r}$ be the matrix with rows $r_1, \dots, r_m$.

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$$\mu(f) := (\min_{c_\alpha \neq 0}\{r_1 \cdot \alpha\}, \dots, \min_{c_\alpha \neq 0}\{r_m \cdot \alpha\}) \in \mathbb{Z}^m.$$

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In $\mathbb{C}[t_1, \dots, t_m][x_1, \dots, x_n]$ the *lift* of f is

$$\tilde{f} := f(\mathbf{t}^{\mathbf{r}\cdot e_1} x_1, \dots, \mathbf{t}^{\mathbf{r}\cdot e_n} x_n)\mathbf{t}^{-\mu(f)} = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha \mathbf{x}^\alpha \mathbf{t}^{\mathbf{r}\cdot\alpha - \mu(f)}.$$

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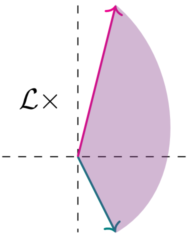
Theorem (B.–Mohammadi–Nájera Chávez)

The lifted ideal $\tilde{J} := (\tilde{f} : f \in J) \subset \mathbb{C}[t_1, \ldots, t_m][x_1, \ldots, x_n]$ is generated by $\{\tilde{g} : g \in \mathcal{G}\}$, where $\mathcal{G}$ is a Gröbner basis for J and C.
Moreover, $\tilde{A}$ is a free $\mathbb{C}[t_1, \ldots, t_m]$-module with standard monomial basis $\mathbb{B}_C$.

Example

Take $f = x_1^2 x_2^2 + x_1^4 - x_2 x_3^3 \in \mathbb{C}[x_1, x_2, x_3]$ and consider in $GF((f))$ the maximal cone C spanned by the rows of $r := \begin{pmatrix} 1 & 4 & 0 \\ 1 & -2 & 0 \end{pmatrix}$ and $\mathcal{L}$.



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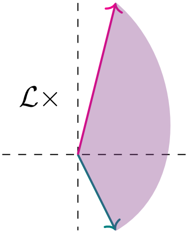
$$\begin{aligned} \tilde{f}(t_1, t_2) &= f(t_1 t_2 x_1, t_1^4 t_2^{-2} x_2, x_3) t_1^{-4} t_2^2 \\ &= t_1^6 x_1^2 x_2^2 + t_2^6 x_1^4 - x_2 x_3^3 \end{aligned}$$



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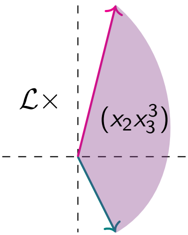

- $\tilde{f}(0, 0) = -x_2x_3^3 = \text{in}_C(f)$,
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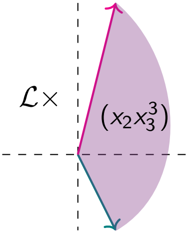

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So $\tilde{A} = \mathbb{C}[t_1, t_2][x_1, x_2, x_3]/(t_1^6x_1^2x_2^2 + t_2^6x_1^4 - x_2x_3^3)$.

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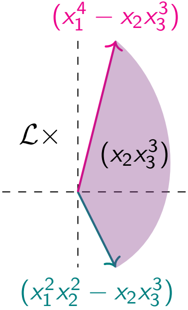

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3. *standard monomial basis $\mathbb{B}_C$ = basis of cluster monomials;*
4. *$C \cap Trop(I_{k,n}) = Trop^+(I_{k,n})$ which is a geometric realization of the cluster complex:*

$$\text{rays of } Trop^+(I_{k,n}) \quad \overset{1:1}{\longleftrightarrow} \quad \text{cluster variables}$$

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4. *$C \cap Trop(I_{k,n}) = Trop^+(I_{k,n})$ which is a geometric realization of the cluster complex:*

$$\text{rays of } Trop^+(I_{k,n}) \quad \overset{1:1}{\longleftrightarrow} \quad \text{cluster variables}$$

$$\text{max cones } \tau_s \in Trop^+(I_{k,n}) \quad \overset{1:1}{\longleftrightarrow} \quad \text{seeds } s \text{ of } A_{k,n}$$

where the toric variety $TV(\Delta(A_{k,n}, g_s))$ is $Proj(A_{\tau_s})$.

[Ilten–Nájera Chávez–Treffinger]: generalized (1) for graded cluster algebras of finte type and (2)/(3) for ADE types.

Arbitrary Grassmannians (in progress)

[GHKK 18]/[Fujita–Oya 20]/[B.–Cheung–Magee–Nájera Chávez]:

s seed of $A_{k,n}$	$\rightsquigarrow$	valuation $g_s : A_{k,n} \setminus \{0\} \to \mathbb{Z}^d$	$\rightsquigarrow$	toric degeneration of Gr(k,n)

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For every seed s of $A_{k,n}$ exists a maximal prime cone τ_s in $\mathrm{Trop}^+(J)$ for an appropriate ideal J with $A_{k,n} \cong k[x_1,\dots,x_m]/J$,

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For every seed s of $A_{k,n}$ exists a maximal prime cone τ_s in $\mathrm{Trop}^+(J)$ for an appropriate ideal J with $A_{k,n} \cong k[x_1, \dots, x_m]/J$, s.t. if J is appropriate for two adjacent seeds s, s' then τ_s and $\tau_{s'}$ share a facet.

Remark: J is *appropriate* for s if $\mathrm{im}(g_s)$ is generated by $g_s(\bar{x}_1), \dots, g_s(\bar{x}_m)$.

Examples

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$$\mathrm{im}(g_s) = \langle g_s(p_{ij}) : 1 \leq i < j \leq n \rangle,$$

so the Plücker ideal is appropriate for all seeds.

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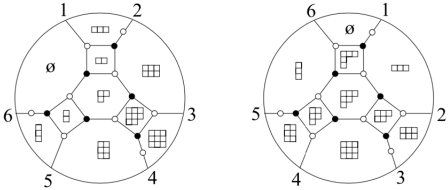

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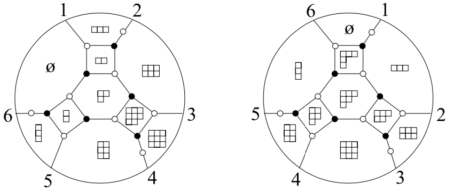

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3. Conjecturally, the ideal I_A of a finite type cluster algebra A is appropriate for all seeds.

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