Cluster varieties with coefficients

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Motivation

- Algebraic: Fomin-Zelevinsky x-pattern and y-pattern. x-pattern can be endowed with coefficients: what about y-pattern?
- Geometric: Gross-Hacking-Keel-Kontsevich introduce flat family $\mathcal{A}_{\text{prin}}$: what is the cluster dual $\mathcal{X}$-analogue?
- Toric degenerations: GHKK degenerate Grassmannians to toric varieties using $\mathcal{A}$-structure. Rietsch-Williams degenerate Grassmannians to toric varieties using $\mathcal{X}$-structure: how are they related?

Mutations

$N \cong \mathbb{Z}^n$ lattice, $\{\cdot,\cdot\} : N \times N \to \mathbb{Z}$, $M = N^*$

$$\begin{aligned} \mu_{(n,m)} : T_N &\dashrightarrow T_N \\ \mu^*_{(n,m)}(z^{m'}) &= z^{m'}(1+z^m)^{m'(n)}. \end{aligned}$$

Let $\{e_i\}$ basis of N, $\{f_i\}$ dual basis of M

$$T_N := \mathrm{Spec}(\mathbb{C}[M]) = \mathrm{Spec}\mathbb{C}[z^{\pm f_1}, \dots, z^{\pm f_n}]$$

$\mathcal{A}$-cluster varieties

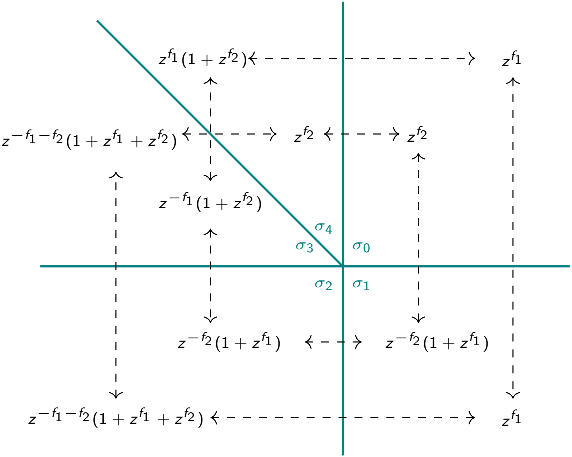
fix $s_0 = \{e_i\}$ basis of N, $v_i := \{e_i, \cdot\} \in M$
$\rightsquigarrow$ $s = \{e_i^s\}$ new basis of N by certain pseudoreflections

Definition ($\mathcal{A}$-cluster mutation)

$$\begin{aligned} \mu_{(-e_k,v_k)} : T_{N,s_0} &\dashrightarrow T_{N,s} \\ \mu^*_{(-e_k,v_k)}(z^{m'}) &= z^{m'}(1+z^{v_k})^{m'(-e_k)} \end{aligned}$$

$$\mathcal{A} := \bigcup_{s\sim s_0} T_{N,s} \text{ glued by } \mathcal{A}\text{-mutations.}$$

Example: $\mathcal{A}$ in case $\mathtt{A}_2$



$\mathcal{X}$-cluster varieties

Exchange M and N

$$\begin{aligned} \mu_{(m,n)} : T_M &\dashrightarrow T_M \\ \mu^*_{(m,n)}(z^{n'}) &= z^{n'}(1+z^n)^{n'(m)}. \end{aligned}$$

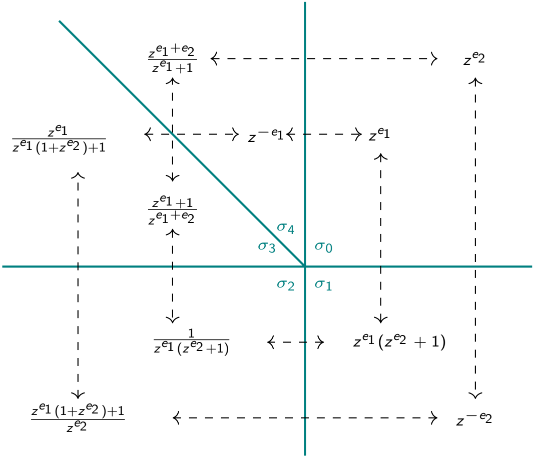
Definition ($\mathcal{X}$-cluster mutation)

$s_0 = \{e_i\}$ basis of N, $v_i := \{e_i, \cdot\} \in M$

$$\begin{aligned} \mu_{(v_k,e_k)} : T_{M,s_0} &\dashrightarrow T_{M,s} \\ \mu^*_{(v_k,e_k)}(z^{n'}) &= z^{n'}(1+z^{e_k})^{n'(v_k)} \end{aligned}$$

$$\mathcal{X} := \bigcup_{s \sim s_0} T_{M,s} \text{ glued by } \mathcal{X}\text{-mutations.}$$

Example: $\mathcal{X}$ in case A_2



Cluster Varieties with coefficients

$R := \mathbb{C}[t_1, \dots, t_n]$, $c \in \mathbb{Z}^n$, $c = c_+ - c_-$

$$\begin{aligned} \mu_{(n,m),c} : T_N(R) &\dashrightarrow T_N(R) \\ \mu^*_{(n,m),c}(\tilde{z}^{m'}) &= \tilde{z}^{m'}(t^{c_+} + t^{c_-}\tilde{z}^m)^{m'(n)} \end{aligned}$$

Definition (cluster mutation with coefficients)

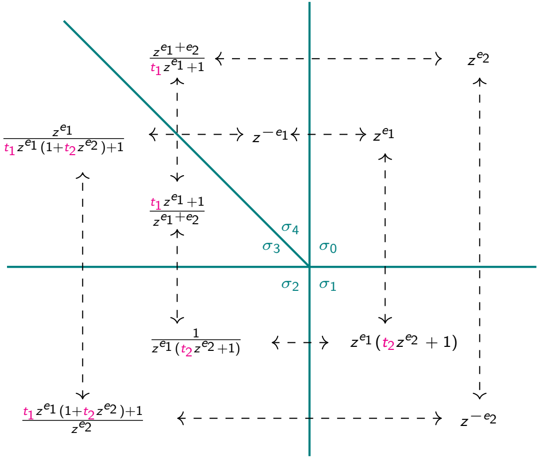
$\mathcal{A}$-cluster mutation with coefficients:

$$\mu_{(-e_k, v_k), c_k} : T_{N,s_0}(R) \dashrightarrow T_{N,s}(R).$$

$\mathcal{X}$-cluster mutation with coefficients:

$$\mu_{(v_k, e_k), c_k} : T_{M,s_0}(R) \dashrightarrow T_{M,s}(R).$$

Example: $\mathcal{X}$ with coefficients in case $\mathtt{A}_2$



Flat families

Lemma

$T_N(R) = T_N \times_{\mathbb{C}} \mathbb{A}^n \to \mathbb{A}^n$ *extends to flat family*

$$\mathcal{A}_{prin} := \bigcup_{s \sim s_0} T_{N,s}(R) \to \mathbb{A}^n$$

$T_M(R) = T_M \times_{\mathbb{C}} \mathbb{A}^n \to \mathbb{A}^n$ *extends to flat family*

$$\mathfrak{X} := \bigcup_{s \sim s_0} T_{M,s}(R) \to \mathbb{A}^n$$

$$\begin{array}{ccccc} \mathcal{A}_{\text{prin}} & & & & \mathfrak{X} \\ \pi_{\mathcal{A}} & \searrow & & \swarrow & \pi_{\mathcal{X}} \\ & & \mathbb{A}^n & & \end{array}$$

g-vectors

Theorem (FZ - Laurent phenomenon)

$\tilde{z}^{f_i^s} \in R[\tilde{z}^{\pm f_1}, \ldots, \tilde{z}^{\pm f_n}]$ *is a global function on* $\mathcal{A}_{prin}$.

Theorem (GHKK)

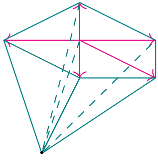
- *In* $\pi_{\mathcal{A}}^{-1}(1) = \mathcal{A}$:
$$\tilde{z}^{f_i^s}|_{t=1} = z^{f_i^s} \in \mathbb{C}[\mathcal{A}],$$
a global function on $\mathcal{A}$.
- *In* $\pi_{\mathcal{A}}^{-1}(0) = T_N$:
$$\tilde{z}^{f_i^s}|_{t=0} = z^{\mathbf{g}_i^s} \in \mathbb{C}[M],$$
a character of T_N.
- *The* $\mathbf{g}_i^s \in M$ *form a simplicial fan called the* **g***-fan.*

$\mathcal{A}$-compactifications

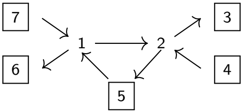
freeze j, i.e. never mutate $\mu_{(-e_j,v_j)}$ $\rightsquigarrow$ allow $z^{f_j}=0$
(Partial) compactification:

$$\overline{\mathcal{A}} \setminus D = \mathcal{A}.$$

$$\begin{array}{c} \overline{\mathcal{A}}_{\mathsf{prin}} \\ \downarrow \pi_{\overline{\mathcal{A}}} \\ \mathbb{A}^n \end{array} \qquad \begin{array}{l} \pi_{\overline{\mathcal{A}}}^{-1}(1) = \overline{\mathcal{A}} \\ \\ \pi_{\overline{\mathcal{A}}}^{-1}(0) = \mathsf{TV}(P) \end{array}$$



Example: Grassmannian



(partially) compactify: $D := D_3 \cup \cdots \cup D_7$

$$\mathcal{A} = \overline{\mathcal{A}} \setminus D \quad \text{and} \quad \overline{\mathcal{A}} = C(\mathrm{Gr}_2(\mathbb{C}^5)),$$

affine cone of $\mathrm{Gr}_2(\mathbb{C}^5) \hookrightarrow \mathbb{P}^6$ with Plücker embedding.

$\mathcal{X}$-compactifications

$z^{e_i^s}$ local function on $\mathcal{X}$ $\rightsquigarrow$ can not allow $z^{e_i^s} = 0$
partially compactify locally: replace T_M by $\mathbb{A}_M = \mathbb{A}^n$

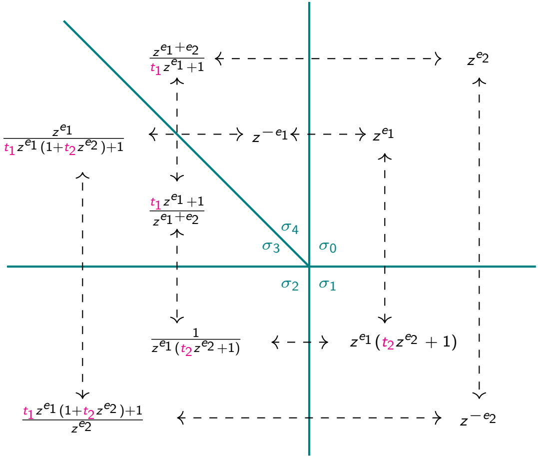
Theorem (BFMN)

$\mathcal{X}$-gluing with coefficients yields a flat family

$$\widehat{\mathfrak{X}} := \bigcup_{s \sim s_0} \mathbb{A}_{M,s}(R) \to \mathbb{A}^n$$

with $\pi_{\widehat{\mathcal{X}}}^{-1}(1) = \widehat{\mathcal{X}}$ Fock-Goncharov special completion, and $\pi_{\widehat{\mathcal{X}}}^{-1}(0) = TV(\mathbf{g}\text{-}fan)$.

Example: $\widehat{\mathcal{X}}$ in case $\mathtt{A}_2$



Homogenization

$R = \mathbb{C}[t_1, \ldots, t_n]$ induces multigrading on $R(\tilde{z}^{e_1}, \ldots, \tilde{z}^{e_n})$:

$$\deg(t_i) := -\deg(z^{e_i}).$$

Theorem (BFMN)

The $\tilde{z}^{e_i^s} \in R(\tilde{z}^{e_1}, \ldots, \tilde{z}^{e_n})$

- are homogeneous of degree $c_i^s \in \mathbb{Z}^n$;
- are unique homogeneous extensions of $z^{e_i^s} \in \mathbb{C}(\tilde{z}^{e_1}, \ldots, \tilde{z}^{e_n})$ of multidegree c_i^s;
- have a limit in $\pi_{\mathcal{X}}^{-1}(0) = T_M$:

$$\lim_{t \to 0} \tilde{z}^{e_i^s} = z^{c_i^s}.$$

Grassmannians: GHKK-degeneration

$$C(\mathrm{Gr}_2(\mathbb{C}^5)) \setminus D = \mathcal{A} \qquad \rightsquigarrow \qquad W_D : \mathcal{X} \to \mathbb{C}$$

$$W_D \quad = \quad \sum_{i=1}^{5} \vartheta_i \in \mathbb{C}[z^{\pm e_1}, \dots, z^{\pm e_7}].$$

Using $\overline{\mathcal{A}}_{\mathrm{prin}}$: degeneration of $\mathrm{Gr}_2(\mathbb{C}^5)$ to toric variety with polytope

$$\Xi_D = \{W_D^{\mathrm{trop}} \geq 0\} \cap H_D \subset \mathbb{R}^6.$$

Grassmannians: RW-degeneration

$\mathrm{Gr}_2(\mathbb{C}^5) \setminus D = \mathcal{A}^\circ$ and $W_{q=1} : \mathcal{A}^\circ \to \mathbb{C}$

$$W_{q=1} = \sum_{i=1}^{5} W_i \in \mathbb{C}[z^{\pm f_1}, \dots, z^{\pm f_7}].$$

Get flat degeneration of $\mathrm{Gr}_2(\mathbb{C}^5)$ to toric variety with polytope

$$\Delta_{q=1} = \{W_{q=1}^{\mathrm{trop}} \geq 0\} \subset \mathbb{R}^6.$$

p-map

$$
\begin{array}{ccc}
C(\mathrm{Gr}_2(\mathbb{C}^5)) \supset \mathcal{A} & \xrightarrow[\sim]{p} & \mathcal{X} \\
\downarrow & & \uparrow \\
\mathrm{Gr}_2(\mathbb{C}^5) \supset \mathcal{A}^\circ & \xrightarrow[\sim]{\bar{p}} & \mathcal{X}^\circ
\end{array}
$$

Consider $w_D : \mathcal{X}^\circ \to \mathbb{C}$.

Key-Lemma (BCMN)

There exists a unique choice of p-map such that

$$\bar{p}^*(w_D) = W_{q=1} \quad \text{and} \quad \bar{p}^*(\Xi_D) = \Delta_{q=1}.$$

The RW-family

$T = (\mathbb{C}^*)^6$ with coordinates $x_{\ydiagram{1}}, x_{\ydiagram{2}}, x_{\ydiagram{3}}, x_{\ydiagram{1,1}}, x_{\ydiagram{2,2}}, x_{\ydiagram{3,3}}$

$$\Phi_S : T \to \mathrm{Gr}_2(\mathbb{C}^5)$$

for example:

$$\begin{aligned}
\Phi_s^*(p_{12}) &= 1, \\
\Phi_s^*(p_{34}) &= x_{\ydiagram{2}} x_{\ydiagram{3}} x_{\ydiagram{2,2}} x_{\ydiagram{3,3}}^2, \\
\Phi_s^*(p_{24}) &= x_{\ydiagram{2}} x_{\ydiagram{3}} x_{\ydiagram{1,1}} x_{\ydiagram{2,2}} x_{\ydiagram{3,3}}^2 (1 + x_{\ydiagram{1}}) \ \ldots
\end{aligned}$$

where $\quad p_{12} = z^{f_7}, \ p_{34} = z^{f_5}, \ p_{24} = z^{-f_2}(z^{f_4+f_1} + z^{f_3+f_5}) \ \ldots$

$x_{\ydiagram{1}}$'s behave as $\mathcal{X}$-variables:

$$\begin{aligned} x_{\ydiagram{2}} &= z^{e_2}, & x_{\ydiagram{1}} &= z^{e_1}, \\ x_{\ydiagram{3,3}} &= z^{-e_3}, & x_{\ydiagram{2,2}} &= z^{-e_2-e_4}, \\ x_{\ydiagram{1,1}} &= z^{-e_1-e_5}, & x_{\ydiagram{3}} &= z^{-e_1-e_2-e_7}. \end{aligned}$$

$\rightsquigarrow$ can extend $\Phi_s^*(p_{ij})$ to $\mathcal{X}$:

$$\begin{aligned} \widetilde{\Phi_s^*(p_{12})} &= 1, \\ \widetilde{\Phi_s^*(p_{34})} &= \tilde{x}_{\ydiagram{2}}\tilde{x}_{\ydiagram{3}}\tilde{x}_{\ydiagram{2,2}}\tilde{x}^2_{\ydiagram{3,3}}, \\ \widetilde{\Phi_s^*(p_{24})} &= \tilde{x}_{\ydiagram{2}}\tilde{x}_{\ydiagram{3}}\tilde{x}_{\ydiagram{1,1}}\tilde{x}_{\ydiagram{2,2}}\tilde{x}^2_{\ydiagram{3,3}}(1+\tilde{t}_{\ydiagram{1}}\tilde{x}_{\ydiagram{1}}) \ \ldots \end{aligned}$$

Recap

GHKK		RW
$\mathcal{A} \subset C(\mathrm{Gr}_2(\mathbb{C}^5))$		$\mathcal{A}^\circ \subset \mathrm{Gr}_2(\mathbb{C}^5)$
$\mathcal{A}_{\mathrm{prin}}$-family	open	$\mathcal{X}$-family
$W_D : \mathcal{X} \to \mathbb{C}$	$\mathcal{A} \cong \mathcal{X}$	$W_{q=1} : \mathcal{A}^\circ \to \mathbb{C}$
$\overline{\mathcal{A}} \setminus D = \mathcal{A}$		$\overline{\mathcal{A}^\circ} \setminus D^\circ = \mathcal{A}^\circ \cong \mathcal{X}^\circ$
$\overline{\mathcal{A}}_{\mathrm{prin}}$-family	$\cong$	$\overline{\mathcal{A}}_{\mathrm{prin}}$-family

Thank you!

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