Newton–Okounkov bodies for cluster varieties

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Valuations

$A = \bigoplus_{i\geq 0} A_i$ a graded k-algebra and domain. A map $\nu : A \setminus \{0\} \to (\mathbb{Z}^d, <)$ is a *(Krull) valuation* if

$$\nu(fg) = \nu(f) + \nu(g), \quad \nu(cf) = \nu(f), \quad \nu(f+g) \geq \min{}_<\{\nu(f), \nu(g)\}$$

for all $f, g \in R \setminus \{0\}$ and $c \in k$.

1. $S(A,\nu) := \mathrm{im}(\nu)$ is the *value semigroup*.
2. ν induces a filtration on A, for $m \in \mathbb{Z}^d$

$$F_m := \{f \in A : \nu(f) \leq m\} \quad \text{and} \quad F_{<m} := \{f \in A : \nu(f) < m\}.$$

3. $\dim(F_m/F_{<m}) \leq 1 \forall m$ [1] $\Rightarrow \mathrm{gr}_\nu(R) \cong k[S(R,\nu)]$
4. $\mathbb{B}$ vector space basis of A is *adapted to ν* if $\mathbb{B} \cap F_m$ is a vector space basis for all m.

[1] e.g. if ν is *full-rank*, i.e. $\mathrm{rank}(S(A,\nu)) = \dim_{\mathrm{Krull}}(A)$ by Abhyankar's inequality

Toric degenerations and the Newton–Okounkov polytope

Theorem (Anderson 2013)

Let $\nu : A \setminus \{0\} \to \mathbb{Z}^d$ be a full-rank valuation with $S(A,\nu)$ finitely generated. Then there exists a toric degeneration of $X = Proj(A)$ to the (not necessarily normal) toric variety $X_0 = Proj(k[S(A,\nu)])$.

X_0 is toric and projective, its normalization $\bar{X}_0$ is defined by the *Newton–Okounkov body*[2] of ν

$$\Delta(A,\nu) := \mathrm{conv}\left(\bigcup_{i>0}\left\{\frac{\nu(f)}{i} : f \in A_i\right\}\right) \subset \mathbb{R}^d.$$

Question: How can we compute $\Delta(A,\nu)$? What are its vertices?

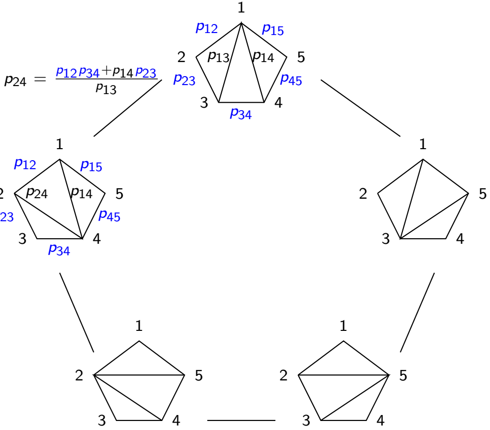
[2] in this case a rational polytope

Grassmannian $\mathrm{Gr}_2(\mathbb{C}^5)$

The homogeneous coordinate ring of $\mathrm{Gr}_2(\mathbb{C}^5)$ with its Plücker embedding:

$$A_{2,5} := \mathbb{C}[p_{ij} : 1 \leq i < j \leq 5]/(p_{ij}p_{kl} - p_{ik}p_{jl} + p_{il}p_{jk})_{1\leq i<j<k<l\leq 5}$$

can be constructed recursively from triangulations of a 5-gon (*seeds*):



This is the prototype of a cluster algebra!

Cluster variety inside $\widetilde{\mathrm{Gr}_2(\mathbb{C}^5)}$

For every seed we get a torus chart: $(\mathbb{C}^*)^7_{p_{13},p_{14},p_{12},p_{15},p_{23},p_{34},p_{45}} \hookrightarrow \widetilde{\mathrm{Gr}_2(\mathbb{C}^5)}$ and they glue along mutations:

$$(\mathbb{C}^*)^7_{p_{13},p_{14},p_{12},\dots,p_{45}} \bigcup_{\mu^*(p_{24})=\frac{p_{12}p_{34}+p_{23}p_{14}}{p_{13}}} (\mathbb{C}^*)^7_{p_{24},p_{14},p_{12},\dots,p_{45}}$$

Recursively we obtain a *cluster variety*

$$\mathcal{A}_{2,5} := \bigcup_{s \text{ triang. of 5-gon}} (\mathbb{C}^*)^7_{p_{ij}:\overline{ij}\in s} \hookrightarrow \widetilde{\mathrm{Gr}_2(\mathbb{C}^5)}$$

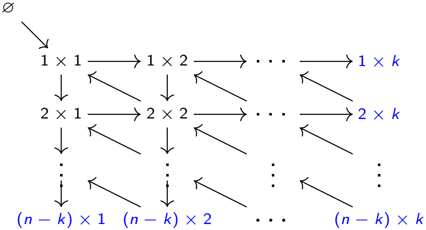
Consider the partial compactification $\overline{\mathcal{A}}_{2,5} := \mathcal{A}_{2,5} \cup \bigcup_{i\in\mathbb{Z}_5}\{p_{i,i+1}=0\}$. Then:

$$\mathcal{O}(\overline{\mathcal{A}}_{2,5}) = A_{2,5} \subset \mathbb{C}[p_{ij}^{\pm 1} : \overline{ij} \in s] \ \forall s \text{ triang. of 5-gon.}$$

Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$

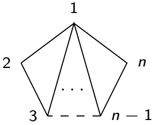
Triangulations and *flips* generalize to quivers and *quiver mutation*:

For a general Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$ seeds are represented by quivers: e.g. $s = (Q, \mathbb{x})$ with $\mathbb{x} = (x_{i\times j})_{i,j}$ where $x_{i\times j} := p_{[1,k-j]\cup[k-j+i+1,k+i]}$ and quiver Q:



Exercise: for $k = 2$

Q corresponds to



g-vectors for cluster algebras

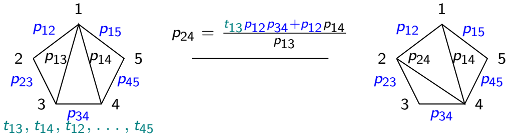
Theorem (Fomin–Zelevinsky 2005)

Given an initial seed $s = (Q, (p_{i\times j})_{i\in[n-k], j\in[k]})$ *of* $A_{k,n}$ *there exists a corresponding cluster algebra with* principal coefficients $A_{k,n}^{prin,\ s} \subset \mathbb{C}[t_{i\times j}][p_{i\times j}^{\pm 1}]_{i\in[n-k],j\in[k]}$ *at* s.
$A_{k,n}^{prin,s}$ *is* M_s*-graded, where* $M_s = \mathbb{Z}^{k(n-k)+1}$ *with basis* $\{f_{i\times j}\}_{i\in[n-k],j\in[k]}$:

$$g_s(p_{i\times j}) = f_{i\times j}, \quad \text{and} \quad g_s(t_{i\times j}) := -\sum \#\{i\times j \to i'\times j'\} f_{i'\times j'}$$

Every cluster variable x *is homogeneous and its degree called* g*-vector.*

Example:



Cluster variety with principal coefficients for $\mathrm{Gr}_2(\mathbb{C}^5)$

Geometrically we obtain a degeneration to a torus

$$\begin{tikzcd} \mathcal{A}_{2,5} \arrow[r, hook] \arrow[d] & \mathcal{A}_{2,5}^{\mathrm{prin,s}} = \bigcup_s (\mathbb{C}^*)^7_s \times \mathbb{A}^7_{t_{13},t_{14},t_{12},\dots,t_{45}} \arrow[d, two heads] & (\mathbb{C}^*)^7 \arrow[l, hook'] \arrow[d] \\ \{1\} \arrow[r, hook] & \mathbb{A}^7 & \{0\} \arrow[l, hook'] \end{tikzcd}$$

Theorem (Gross-Hacking-Keel-Kontsevich)

The cluster variety $\mathcal{A}_{k,n}^{\mathrm{prin,s}}$ with principal coefficients at a seed s induces a toric degeneration of the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$. Moreover, Fomin–Zelevinsky's g-vectors are characters of the torus in the central fibre.

Holds more generally for partial compactifications of cluster varieties that satisfy the full Fock–Goncharov conjecture.

g-vector valuation

Proposition (GHKK, Fujita–Oya, B–Cheung–Magee–Nájera Chávez)

Let s be an arbitrary seed of $A_{k,n}$ and x denote any cluster variable, then $x \mapsto g_s(x)$ extends to a (full-rank homogeneous) valuation with finitely generated value semigroup:

$$g_s : A_{k,n} \setminus \{0\} \to M_s \cong \mathbb{Z}^{k(n-k)+1} \quad \text{with} \quad x \mapsto g_s(x)$$

that defines the $\mathcal{A}_{k,n}^{\mathrm{prin},s}$-toric degeneration of $\mathrm{Gr}_k(\mathbb{C}^n)$. Moreover, $A_{k,n}$ has a $\mathbb{C}$-basis adapted to all $g_{s'}$ *simultaneously* called the ϑ-basis.

Remark: The Proposition holds more generally for any cluster algebra that satisfies the *full Fock–Goncharov conjecture*.

Newton–Okounkov bodies for Grassmannains

Proposition

For every seed s of $A_{2,n}$ the value semigroup $S(A_{2,n}, g_s)$ is generated by the g-vectors of Plücker coordinates and its Newton–Okounkov body is

$$\Delta(A_{2,n}, g_s) = \mathrm{conv}(g_s(p_{ij}) : 1 \leq i < j \leq n).$$

Theorem (B.–Cheung–Magee–Nájera Chávez)

For arbitrary $\mathrm{Gr}_k(\mathbb{C}^n)$ Rietsch–Williams define a valuation $v_s : A_{k,n} \to \mathbb{Z}^{k(n-k)}$ for every plabic graph s (or more generally for every seed s of $A_{k,n}$). We can show that

$$\Delta(A_{k,n}, g_s) \cong \Delta(A_{k,n}, v_s).$$

Connections to Gröbner theory

For $A_{2,n}, A_{3,6}, A_{3,7}, A_{3,8}$ the ϑ-basis consists of all monomials in cluster variables of the same seed, called *cluster monomials*.

Proposition (B.–Mohammadi–Nájera Chávez)

For $(k, n) \in \{(2, n), (3, 6)\}$ the ϑ-basis of $A_{k,n}$ is a standard monomial basis associated to a maximal cone C in the Gröbner fan of an ideal $J_{k,n}$ representing $A_{k,n}$.
Moreover, every $k(n-k)+1$-dimensional face of C lies inside the tropicalization of $J_{k,n}$ and induces a toric degeneration of $\mathrm{Gr}_k(\mathbb{C}^n)$ whose central fibre is

$$TV(\Delta(A_{k,n}, g_s)).$$

Tropical cluster dual $\mathcal{X}$-variety

There exists a tropical cluster variety $\mathcal{X}_{2,5}^{\text{trop}}(\mathbb{Z}) := \bigcup_{s \text{ triang. of 5-gon}} M_s$ where $M_s = \mathbb{Z}^7$ with free generating set $\{f_{ij}, \overline{ij} \in s\}$ and glued along *bijections* defined by certain shearing:

$$M_{\{f_{13},f_{14},f_{12},\dots,f_{45}\}} \quad \underset{\substack{f_{24}=f_{23}+f_{14}-f_{13} \\ T_{13}(m)=m+\max\{m_{13},0\}(f_{12}+f_{34}-f_{23}-f_{14})}}{\bigcup} \quad M_{\{f_{24},f_{14},f_{12},\dots,f_{45}\}}$$

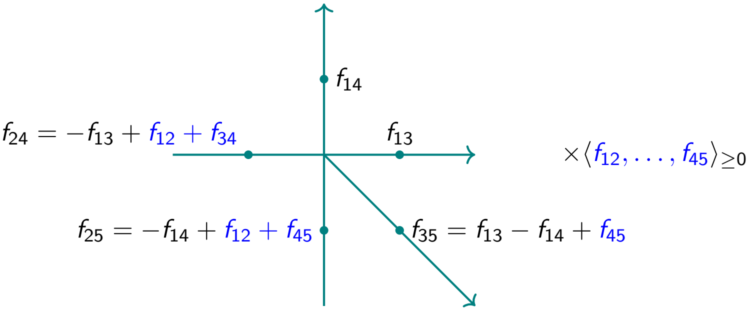
For each s we may identify *non-canonically* $\mathcal{X}_{2,5}^{\text{trop}}(\mathbb{Z}) \equiv_s M$.

[GHKK]/[Marsh–Scott]/[Shen–Weng] Elements of the ϑ-basis for $A_{k,n}$ are indexed by points in a "cone" $\Xi \subset \mathcal{X}_{k,n}^{\text{trop}}(\mathbb{Z})$:

$$\begin{matrix} (\overline{\mathcal{A}}_{k,n}, D) & & (\mathcal{X}_{k,n}, W : \mathcal{X}_{k,n} \to \mathbb{C}) \\ \vartheta\text{-basis of } A_{k,n} & \longleftrightarrow & \Xi := \{W^{\text{trop}}(x) \geq 0\} \subset \mathcal{X}_{k,n}^{\text{trop}}(\mathbb{Z}) \end{matrix}$$

Wall&chamber structure and the g-fan

Pulling back the positive orthants of each copy of $M_{\mathbb{R}}$ along the shears T'_{ij}s yields a *wall and chamber structure* in $\mathcal{X}_{2,5}^{\text{trop}}(\mathbb{R})$:



It contains a full-dimensional simplicial fan known as the *g-fan*:

$$\begin{aligned} \text{maximal simplicial cones} &\quad\leftrightarrow\quad \text{seeds} \\ \text{primitive ray generators} &\quad\leftrightarrow\quad g\text{-vectors of cluster variables} \end{aligned}$$

NO bodies for compactificatins of cluster varieties

Theorem (B.–Cheung–Magee–Nájera Chávez)

There exists a "convex" set $\Delta_{BL}(A_{k,n}) \subset \mathcal{X}_{k,n}^{trop}(\mathbb{R})$ independent of s:

$$\Delta_{BL}(A_{k,n}) \equiv_s \Delta(A_{k,n}, g_s).$$

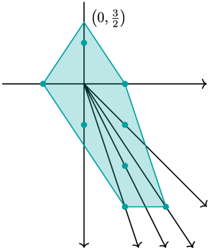
$$\begin{tikzcd}
& \Delta_{BL}(A_{k,n}) \arrow[dl, "\equiv_s"'] \arrow[dr, "\equiv_{s'}"] & \\
\Delta(A_{k,n}, g_s) \arrow[rr, "\text{piece-wise lineas}", "\text{``wall-crossing''}"'] & & \Delta(A_{k,n}, g_{s'})
\end{tikzcd}$$

Proposition (Escobar–Harada, B.–Mohammadi–Nájera Chávez)

The piecewise-linear maps between two Newton–Okounkov polytopes $\Delta(A_{2,n}, g_s)$ and $\Delta(A_{2,n}, g_{s'})$ coincide with Escobar–Harada's *algebraic wall-crossing* for Newton–Okounkov polytopes arising from adjacent maximal prime cones in the tropicalization of $\mathrm{Gr}_2(\mathbb{C}^n)$.

Broken line convexity

In $\mathcal{X}_{k,n}^{\text{trop}}(\mathbb{R})$ we don't
have straight lines, but piece-wise linear *broken lines*.



[Cheung–Magee–Nájera Chávez] introduce
broken line convexity: a closed set $S \subset \mathcal{X}_{k,n}^{\text{trop}}(\mathbb{R})$
is broken line convex iff $\forall a, b \in S$ and any
broken line segment ℓ between a, b we have $\ell \subset S$.

Lemma (Cheung–Magee–Nájera Chávez)

Under the identification $\mathcal{X}_{k,n}^{trop}(\mathbb{R}) \equiv_s M_{\mathbb{R}}$ every broken line convex set is a convex set.

Intrinsic Newton–Okounkov body

For $f \in A_{k,n}$ we have $f = \sum c_m \vartheta_m$. Define

$$\mathrm{New}_\vartheta(f) := \mathrm{conv}_{BL}(m \in \mathcal{X}_{k,n}^{\mathrm{trop}}(\mathbb{Z}) : c_m \neq 0) \subset \mathcal{X}_{k,n}^{\mathrm{trop}}(\mathbb{R})$$

Then the *intrinsic Newton–Okounkov body is*

$$\Delta_{BL}(A_{k,n}) = \mathrm{conv}_{BL}\left(\bigcup_{i \geq 1} \frac{\mathrm{New}_\vartheta(f)}{i} : f \in (A_{k,n})_i \right).$$

Corollary (B.–Cheung–Magee–Nájera Chávez)

For every seed s we have

$$\Delta(A_{k,n}, g_s) = \mathrm{conv}_{BL}\left(g_s(p_I) : I \in \binom{[n]}{k} \right)$$

In particular, $\Delta(A_{k,n}, g_s)$ is a rational polytope with integral vertices of form $g_s(p_I)$, and (depending on s) additional rational vertices in walls.

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