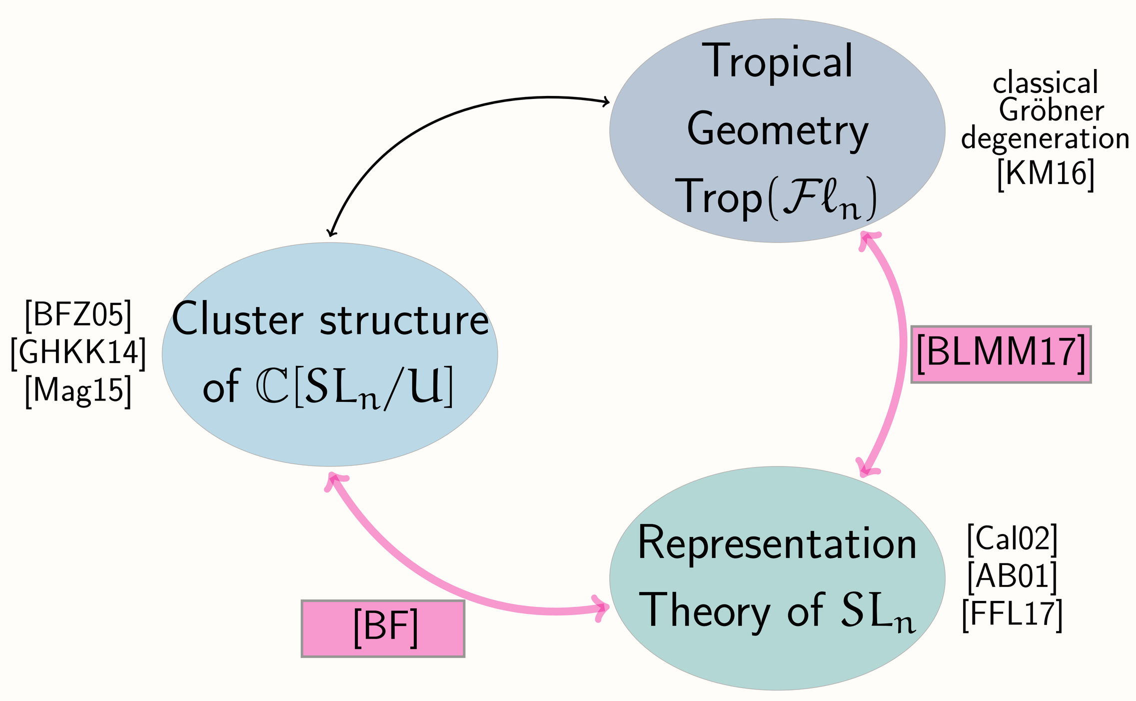
Toric degenerations of flag varieties:
tropical geometry, representation theory and cluster algeras

Lara Bossinger

University of Cologne, Germany

Motivation

A *toric degeneration* of a variety X is flat family $\pi : \mathcal{X} \to \mathbb{A}^n$ with generic fibre isomorphic to X and central fibre isomorphic to a toric varitey Y. They arise in various ways, for example, for the full flag varity $\mathcal{F}\ell_n = SL_n(\mathbb{C})/B$ in:



If isomorphic toric degenerations arise from different constructions this can be interpretet as a "*combinatorial shadow*" of a deeper connection between the theories behind. As a first step it is therefore necessary to compare the toric degenerations each construction yields.

Tropical Geometry

Using the Plücker embedding $\mathrm{Gr}(k,n) \hookrightarrow \mathbb{P}^{\binom{n}{k}-1}$ for Grassmannians we fix the embedding

$$\mathcal{F}\ell_n \hookrightarrow \mathrm{Gr}(1,n)\times\cdots\times\mathrm{Gr}(n-1,n) \hookrightarrow \mathbb{P}^{\binom{n}{1}-1}\times\cdots\times\mathbb{P}^{\binom{n}{n-1}-1}.$$

As a result we obtain an ideal $I_n \subset \mathbb{C}[p_I \mid I \subset \{1,\dots,n\}]$ with $V(I_n) = \mathcal{F}\ell_n$ and I_n is generated by Plücker relations, e.g. $I_3 = \langle p_1p_{23} - p_2p_{13} + p_3p_{12}\rangle$.

Definition: Let $I \subset \mathbb{C}[x_1,\dots,x_n]$ be an ideal and $f = \sum a_u x^u \in I$. We define with respect to $\mathbf{w} \in \mathbb{R}^n$

- the *initial form of* f as $\mathrm{in}_\mathbf{w}(f) = \sum_{\mathbf{w}\cdot u \text{ minimal}} a_u x^u$,
- the *initial ideal of* I as $\mathrm{in}_\mathbf{w}(I) = \langle \mathrm{in}_\mathbf{w}(f) \mid f \in I\rangle$.

Example: Take $I_3 \subset \mathbb{C}[p_1,p_2,p_3,p_{12},p_{13},p_{23}]$ and $\mathbf{w} = (0,0,1,0,0,0) \in \mathbb{R}^6$. Then

$$\mathrm{in}_\mathbf{w}(p_1p_{23} - p_2p_{13} + p_3p_{12}) = p_1p_{23} - p_2p_{13}.$$

Let $X = V(I)$ for $I \subset \mathbb{C}[x_1,\dots,x_n]$ and $\mathbf{w} \in \mathbb{R}^n$ arbitrary. Then we have a flat family $\pi : \mathcal{X} \to \mathbb{A}^1$ with

$$\pi^{-1}(t) \cong V(I) \text{ for } t \neq 0, \text{ and } \pi^{-1}(0) \cong V(\mathrm{in}_\mathbf{w}(I)).$$

If $\mathrm{in}_\mathbf{w}(I)$ is binomial and prime, then $V(\mathrm{in}_\mathbf{w}(I))$ is a toric variety. Hence, the flat family defines a *(Gröbner) toric degeneration* of X.

Definition: The *tropicalized flag variety* is defined as

$$\mathrm{Trop}(\mathcal{F}\ell_n) = \left\{ w \in \mathbb{R}^{\binom{n}{1}+\dots+\binom{n}{n-1}} \;\middle|\; \begin{array}{c}\mathrm{in}_w(I_n) \text{ contains}\\ \text{no monomials}\end{array} \right\}$$

$\mathrm{Trop}(\mathcal{F}\ell_n)$ has the structure of a polyhedral fan: for $\mathbf{w}, \mathbf{w}'$ in relative interior of a cone C we have

$$\mathrm{in}_\mathbf{w}(I_n) = \mathrm{in}_{\mathbf{w}'}(I_n) =: \mathrm{in}_C(I_n).$$

A cone C is called *prime*, if $\mathrm{in}_C(I_n)$ is a prime ideal.
The S_n-action on $\mathcal{F}\ell_n$, for $\sigma \in S_n$ induced by

$$p_{\{i_1,\dots,i_k\}} \mapsto \mathrm{sgn}(\sigma)p_{\{\sigma(i_1),\dots,\sigma(i_k)\}},$$

and the $\mathbb{Z}^2$-action induced by $p_I \mapsto p_{[n]\setminus I}$ extend to $\mathrm{Trop}(\mathcal{F}\ell_n)$.

Aim: Find (up to symmetry) all maximal prime cones $C \subset \mathrm{Trop}(\mathcal{F}\ell_n)$ (i.e. binomial and prime $\mathrm{in}_C(I_n)$).
In [KM16] Kaveh and Manon give a construction of a full rank valuation ν_C associated with (maximal) prime cone C. One of their main results is that the associated graded alegbra $\mathrm{gr}_\nu(\mathbb{C}[\mathcal{F}\ell_n])$ (obtained from the filtration induced by ν_C) is isomorphic to $\mathbb{C}[p_I]/\mathrm{in}_C(I_n)$. They further construct the Newton–Okounkov polytopes NO_C that are the polytopes associated to the normalization of the toric variety $V(\mathrm{in}_C(I_n))$.

Theorem [BLMM17]
For $\mathcal{F}\ell_4$ there are 78 maximal cones in $\mathrm{trop}(\mathcal{F}\ell_4)$ grouped in five $S_4 \times \mathbb{Z}^2$-symmetry classes.

Orbit	Size	Prime	F-vector of NO_C
1	24	yes	(42, 141, 202, 153, 63, 13)
2	12	yes	(40, 132, 186, 139, 57, 12)
3	12	yes	(42, 141, 202, 153, 63, 13)
4	24	yes	(43, 146, 212, 163, 68, 14)
5	6	no	Not applicable

Representation Theory

Toric degenerations of flag varieties have been studied intensively in representation theory over the last two decades (see [FFL16] for an overview). Two examples we want to focus on are:

1. String polytopes (defined by Littelmann resp. Berenstein-Zelevinsky, degeneration due to Caldero)
2. FFLV polytope (definition and degeneration due to Feigin-Fourier-Littelmann, existence conjectured by Vinberg)

Both can be realized as NO-polytopes (see [Kav15], resp. [Kir15]). Therefore, we want to compare them to the NO-polytopes from $\mathrm{Trop}(\mathcal{F}\ell_n)$.

For $\mathcal{F}\ell_4$ up to isomorphism there are four classes of string polytopes and one FFLV polytope. In [BLMM17] we compare those polytopes using `polymake`:

Orbit	Combinatorially equivalent polytopes
1	String 2
2	String 1 (Gelfand-Tsetlin)
3	String 3 and FFLV
4	-

Cluster Algebras

Idea: start with set of algebraically independent generators (*seed*) for $\mathbb{C}[SL_n/U]$ and use *mutation* to sucessively generate all seeds.

Example: For $\mathbb{C}[SL_4/U]$ choose as initial seed

$$s_0 = \{p_2, p_3, p_{23}, p_1, p_{12}, p_{123}, p_4, p_{34}, p_{234}\}.$$

Start replacing pink ones (one at a time) by others using mutation, e.g.

$$p_{13} = \frac{p_1p_{23} + p_3p_{12}}{p_2}.$$

Then $\mu_2(s_0) = \{p_{13}, p_3, p_{23}, p_1, p_{12}, p_{123}, p_4, p_{34}, p_{234}\}$.

In [GHKK14] the authors give a construction of toric degenrations that can be applied as follows: for every seed s in $\mathbb{C}[SL_n/U]$ the superpotential $W|_s$ is a Laurent polynomial in the variables in s. Tropicalizing $W|_s$ (and intersecting with certain hyperplanes H_λ) yields a polytope

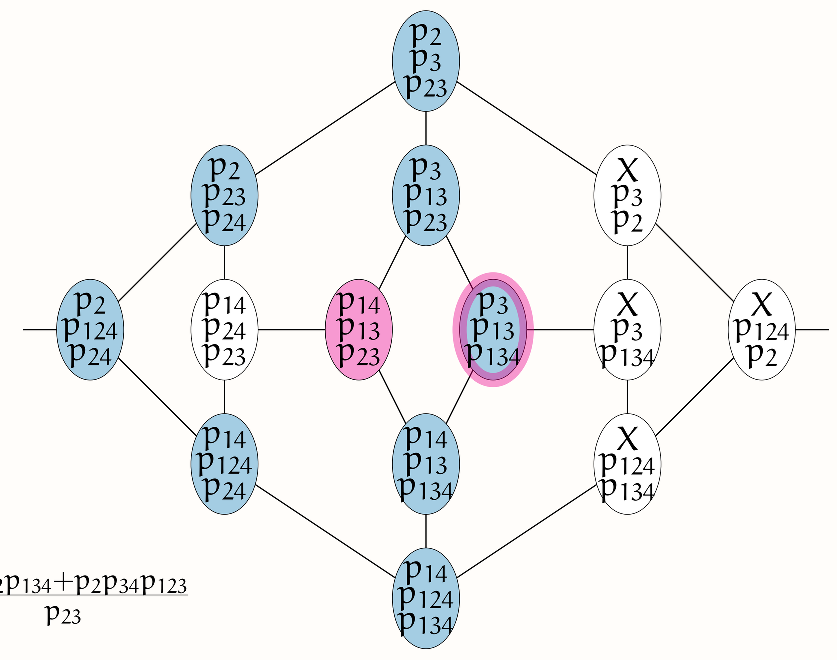
$$\Xi_s(\lambda) = \{W|_s^{\mathrm{trop}}(x) \geq 0\} \cap H_\lambda \subset \mathbb{R}^{\frac{n^2+n}{2}},$$

that is associated to the toric variety degenerating $\mathcal{F}\ell_n$. A natural question to ask is whether the GHKK-construction specializes to string polytopes.

Theorem [BF]
For every string polytope there exists a unique seed s such that the string polytope is unimodularly equivalent to the polytope $\Xi_s(\lambda)$.

For example, if $n = 4$, the string polytopes (resp. FFLV polytope) can be found in the mutation graph of $\mathbb{C}[SL_4/U]$ up to unimodular (resp. combinatorial) equivalence as follows:

$$X = \frac{p_3p_{12}p_{134} + p_2p_{34}p_{123}}{p_{23}}$$



References

[BF] Lara Bossinger and Ghislain Fourier. String cone and superpotential combinatorics for flag and schubert varieties in type A. *arXiv preprint arXiv:1611.06504*.

[BFZ05] Arkady Berenstein, Sergey Fomin, and Andrei Zelevinsky. Cluster algebras. III. Upper bounds and double Bruhat cells. *Duke Math. J.*, 126(1):1–52, 2005.

[BLMM17] Lara Bossinger, Sara Lamboglia, Kalina Mincheva, and Fatemeh Mohammadi. Computing toric degenerations of flag varieties. *In: Smith G., Sturmfels B. (eds) CAG. Fields Inst. Comm.*, 80:247–281, 2017.

[FFL16] Xin Fang, Ghislain Fourier, and Peter Littelmann. On toric degenerations of flag varieties. *Representation Theory - Current Trends and Perspectives, EMS Series of Congress Reports*, pages 187–232, 2016.

[FFL17] Xin Fang, Ghislain Fourier, and Peter Littelmann. Essential bases and toric degenerations arising from birational sequences. *Adv. Math.*, 312:107–149, 2017.

[GHKK14] Mark Gross, Paul Hacking, Sean Keel, and Maxim Kontsevich. Canonical bases for cluster algebras. *arXiv preprint arXiv:1411.1394*, 2014.

[Kav15] Kiumars Kaveh. Crystal bases and Newton–Okounkov bodies. *Duke Mathematical Journal*, 164(13):2461–2506, 2015.

[Kir15] Valentina Kiritchenko. Newton–Okounkov polytopes of flag varieties. *Transformation Groups*, pages 1–16, 2015.

[KM16] Kiumars Kaveh and Christopher Manon. Khovanskii bases, higher rank valuations and tropical geometry. *arXiv preprint arXiv:1610.00298*, 2016.

[Mag15] Timothy Magee. Fock-Goncharov conjecture and polyhedral cones for $U \subset SL_n$ and base affine space SL_n/U. *arXiv preprint arXiv:1502.03769*, 2015.

Geometry and Representation Theory of Algebraic Groups, 5 - 9 March 2018, Bad Honnef, Germany