# Families of Gröbner degenerations

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# Overview

1. Motivation
2. Review on Gröbner theory
3. Construction and Main Theorem
4. Application I: universal coefficients for cluster algebras
5. Application II: Toric families

# Motivation

Understand how different toric degenerations of a projective variety are related.

*Slogan: Knowing all possible toric degenerations of a variety is equivalent to knowing its mirror dual variety.*

Today: understand those toric degenerations of a polarized projective variety that *"share a common basis"*.

# Initial ideals

Let $f = \sum c_\alpha \mathbf{x}^\alpha \in \mathbb{C}[x_1, \dots, x_n]$ with $c_\alpha \in \mathbb{C}$, $\alpha \in \mathbb{Z}^n_{\geq 0}$ and $\mathbf{x}^\alpha := x_1^{\alpha_1} \cdot \dots \cdot x_n^{\alpha_n}$.

For $w \in \mathbb{R}^n$ we define its *initial form with respect to $w$* as

$$\mathrm{in}_w(f) := \sum_{w \cdot \beta = \min_{c\alpha \neq 0}\{w \cdot \alpha\}} c_\beta \mathbf{x}^\beta.$$

For $J \subset \mathbb{C}[x_1, \dots, x_n]$ an ideal we define its *initial ideal with respect to $w$* as $\mathrm{in}_w(J) := \langle \mathrm{in}_w(f) \ : \ f \in J \rangle$.

**Example**

For $f = x_1 x_2^2 + x_1^2 + x_2 \in \mathbb{C}[x_1, x_2]$ and $w = (1, 0)$ we compute

$$\mathrm{in}_w(f) = x_2.$$

# Gröbner fan and Gröbner degenerations

**Definition**

For an ideal $J \subset \mathbb{C}[x_1, \ldots, x_n]$ its *Gröbner fan* $GF(J)$ is $\mathbb{R}^n$ with fan structure defined by

$$v, w \in C^\circ \quad \Leftrightarrow \quad \mathrm{in}_v(J) = \mathrm{in}_w(J).$$

Notation: $\mathrm{in}_C(J) := \mathrm{in}_w(J)$ for any $w \in C^\circ$.

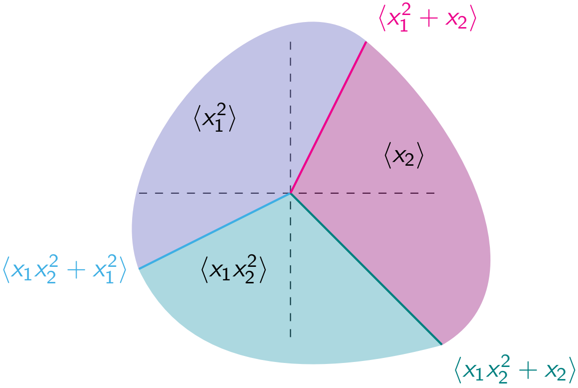
Every open cone $C^\circ \in GF(J)$ defines a *Gröbner degeneration*

$$\pi : \mathcal{V} \to \mathbb{A}^1$$

with $\pi^{-1}(t) \cong V(J)$ for $t \neq 0$ and $\pi^{-1}(0) = V(\mathrm{in}_C(J))$.

# Example

Take $I = \langle x_1x_2^2 + x_1^2 + x_2 \rangle \subset \mathbb{C}[x_1, x_2]$. Then $GF(I)$ is $\mathbb{R}^2$ with the fan structure:



# Standard monomial basis

Let $A := \mathbb{C}[x_1, \dots, x_n]/J$ and $A_\tau := \mathbb{C}[x_1, \dots, x_n]/\mathrm{in}_\tau(J)$ for $\tau \in GF(J)$.

Fix a maximal cone $C \in GF(J)$, then the ideal $\mathrm{in}_C(J)$ is generated by monomials. For every face $\tau \subseteq C$ we define

$$\mathbb{B}_{C,\tau} := \{\bar{\mathbf{x}}^\alpha \in A_\tau \mid \mathbf{x}^\alpha \notin \mathrm{in}_C(J)\}.$$

Then $\mathbb{B}_{C,\tau}$ is a vector space basis for $A_\tau$ called *standard monomial basis*.

In particular, $\mathbb{B}_C := \mathbb{B}_{C,\{0\}}$ is a vector space basis for $A = A_{\{0\}}$.

$\rightsquigarrow$ *All degenerations* $\{V(\mathrm{in}_\tau(J)) : \tau \subseteq C\}$ share one *standard monomial basis!*

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Then $\mathbb{B}_{C,\tau}$ is a vector space basis for $A_\tau$ called *standard monomial basis*[1].

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[1] Due to Lakshmibai–Seshadri, generalized by Sturmfels–White

# Family of ideals

Let $C \in GF(J)$ be a maximal cone and choose $r_1, \dots, r_m$ ray generators of $C$. Let $\mathbf{r}$ be the matrix with rows $r_1, \dots, r_m$. Define for $f = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha \mathbf{x}^\alpha \in J$ its *lift*

$$\tilde{f} := f(\mathbf{t}^{\mathbf{r}\cdot e_1} x_1, \dots, \mathbf{t}^{\mathbf{r}\cdot e_n} x_n) \prod_{i=1}^{m} t_i^{-\min\{r_i \cdot \alpha | c_\alpha \neq 0\}}.$$

**Definition/Proposition**

We define the *lifted ideal*

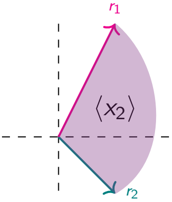
$$J_C(\mathbf{t}) := \langle \tilde{g}_1, \dots, \tilde{g}_s \rangle \subset \mathbb{C}[t_1, \dots, t_m][x_1, \dots, x_n]$$

where $\{g_1, \dots, g_s\}$ is a reduced Gröbner basis for $J$ and $C$.

# Example

Take $f = x_1x_2^2 + x_1^2 + x_2 \in \mathbb{C}[x_1, x_2]$ and consider in $GF(\langle f \rangle)$ the maximal cone $C$ spanned by $r_1 := (\,1\ 2\,)$ and $r_2 := (\,1\ {-1}\,)$. We compute

$$\begin{aligned} \tilde{f}(t_1, t_2) &= f(t_1t_2x_1, t_1^2t_2^{-1}x_2)t_1^{-2}t_2^1 \\ &= t_1^3x_1x_2^2 + t_2^3x_1^2 + x_2 \end{aligned}$$




- $\tilde{f}(0, 0) = x_2 = \mathrm{in}_C(f)$,
- $\tilde{f}(1, 0) = x_1x_2^2 + x_2 = \mathrm{in}_{r_1}(f)$,
- $\tilde{f}(0, 1) = x_1^2 + x_2 = \mathrm{in}_{r_2}(f)$,
- $\tilde{f}(1, 1) = f$.

# Main result

Let $\mathcal{A}_C := \mathbb{C}[t_1, \ldots, t_m][x_1, \ldots, x_n]/J_C(\mathbf{t})$.

**Theorem (B.–Mohammadi–Nájera Chávez)**

*The algebra $\mathcal{A}_C$ is a flat $\mathbb{C}[t_1, \ldots, t_m]$-module. Moreover,*

$$\pi_C : Spec(\mathcal{A}_C) \to \mathbb{A}^m$$

*is a flat family with generic fiber $V(J)$ and special fibers isomorphic to $V(in_\tau(J))$ for every face $\tau \subseteq C$.*

**Example**

$$\mathcal{A}_C = \mathbb{C}[t_1, t_2][x_1, x_2]/\langle t_1^3 x_1 x_2^2 + t_2^3 x_1^2 + x_2 \rangle.$$

# Toric degenerations

$GF(J)$ contains a subfan of dimension $\dim_{\text{Krull}} A$ called the *tropicalization of J*

$$\text{Trop}(J) := \{w \in \mathbb{R}^n \mid \text{in}_w(J) \not\ni \text{monomials}\}.$$

**Corollary (B.–Mohammadi–Nájera Chávez)**

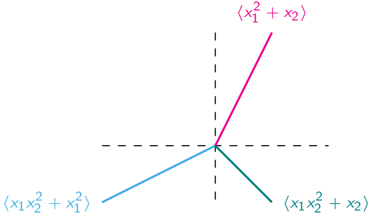
*Consider the fan $\Sigma := C \cap Trop(J)$. If there exists $\tau \in \Sigma$ with $in_\tau(J)$ binomial and prime, then the family*

$$\pi_C : Spec(\mathcal{A}_C) \to \mathbb{A}^m$$

*contains toric fibers isomorphic to $V(in_\tau(J))$.*

# Example

For $J = \langle x_1x_2^2 + x_1^2 + x_2\rangle \subset \mathbb{C}[x_1, x_2]$ the tropicalization $\mathrm{Trop}(J)$ consists of three one-dimensional cones:




For $C = \langle r_1, r_2\rangle$ we have $\Sigma = \langle r_1\rangle \cup \langle r_2\rangle$ and $V(\mathrm{in}_{\langle r_1\rangle}(J))$ is toric.

# Application I: universal coefficients for cluster algebras

A *cluster algebra*[2] $A \subset \mathbb{C}(x_1, \ldots, x_n)$ is a commutative ring defined recursively by

1. *seeds*: maximal sets of algebraically independent algebra generators,
   its elements are called *cluster variables*;
2. *mutation*: an operation to create a new seed from a given one by replacing one element.

[2] Defined by Fomin–Zelevinsky.

# Application I: Grassmannians

Consider the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$ with Plücker embedding[3]. Then its homogeneous coordinate ring

$$A_{k,n} = \mathbb{C}\left[p_J \mid J = \{j_1, \dots, j_k\} \subset [n]\right] / I_{k,n}$$

is a cluster algebra [Scott06].

$k \leq 2$ Plücker coordinates $=$ cluster variables.

$k \geq 3$ Plücker coordinates $\subsetneq$ cluster variables.

$k = 2$ or $k = 3, n \in \{6, 7, 8\}$ finitely many seeds;

otherwise infinitely many seeds.

[3] Assume $k \leq \lfloor \frac{n}{2} \rfloor$.

# Application I: toric degenerations

Fix a seed $s$, then $A$ can be endowed with *principal coefficients at the seed s*

$$A_s^{\mathrm{prin}} \subset \mathbb{C}[t_1, \dots, t_n](x_1, \dots, x_n).$$

Under some technical assumptions:

1. $A_s^{\mathrm{prin}}$ has a $\mathbb{C}[t_1, \dots, t_n]$-basis called *$\vartheta$-basis*[4], which is independent of $s$;
2. if $A$ is the homogeneous coordinate ring of a projective variety $X$ then $A_s^{\mathrm{prin}}$ defines a toric degeneration of $X$ to $X_{s,0}$.

$\rightsquigarrow$ *all these degenerations share the $\vartheta$-basis!*

[4]Due to Gross–Hacking–Keel–Kontsevich.

# Application I: universal coefficients

Now assume $A$ has finitely many seeds.

Algebraically, we can endow $A$ with *universal coefficients*:

$$A^{\mathrm{univ}} \subset \mathbb{C}[t_1, \dots, t_{\#\mathrm{cv}}](x_1, \dots, x_n),$$

where $\#$cv is the number of cluster variables.
Moreover, we have a unique *specialization map* for every seed $s$:

$$A^{\mathrm{univ}} \to A_s^{\mathrm{prin}}.$$

- ⊕ $A^{\mathrm{univ}} \leftrightsquigarrow$ all toric degenerations $X_{s,0}$,
- ⊖ $A^{\mathrm{univ}}$ is defined only *recursively*.

# Application I: Grassmannians

Consider $\mathrm{Gr}_k(\mathbb{C}^n)$ for $(k, n) \in \{(2, n), (3, 6)\}$.
Then there exists a presentation

$$A_{k,n} \cong \mathbb{C}[\text{cluster variables}]/J_{k,n}.$$

$\mathrm{Gr}_2(\mathbb{C}^n)$: $\{\text{cluster variables}\} = \{\text{Plücker coordinates}\}$, and $J_{2,n} = I_{2,n}$;

$\mathrm{Gr}_3(\mathbb{C}^6)$: $\{\text{cluster variables}\} = \{\text{Plücker coordinates}, x, y\}$, and eliminating $x$ and $y$ from $J_{3,6}$ gives $I_{3,6}$.

# Application I: connection to our work

**Theorem (B.–Mohammadi–Nájera Chávez)**

*There exists a unique maximal cone $C$ in the Gröbner fan of $J_{k,n}$ such that*

1. *we have a canonical isomorphism $\mathcal{A}_C \cong A_{k,n}^{\mathrm{univ}}$;*
2. *the standard monomial basis $\mathbb{B}_C$ coincides with the $\vartheta$-basis;*
3. *for every maximal cone $\tau \in C \cap Trop(J_{k,n})$ the variety $V(in_\tau(J_{k,n}))$ is toric.*

# Application II: Toric families

Let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be a homogeneous prime ideal.
Assume there exits $\sigma_1, \ldots, \sigma_s$ maximal cones in Trop$(I)$ with

- $\text{in}_{\sigma_i}(I)$ is toric for all $i$,
- $\sigma_1, \ldots, \sigma_s$ are faces of one maximal cone $C$ in $GF(I)$.

Denote by $\Sigma$ the fan with maximal cones $\sigma_1, \ldots, \sigma_s$.

**Theorem (Kaveh–Manon)**

*There exists a toric family $\psi_\Sigma : \mathbf{Spec}(\mathcal{R}_\Sigma) \to TV(\Sigma)$ with generic fiber $V(I)$ and special fibers $V(in_{\sigma_i}(I))$ over every torus fixed point.*

Here $\mathcal{R}_\Sigma$ is a flat sheaf of Rees algebras on $TV(\Sigma)$ and $\mathbf{Spec}(\mathcal{R}_\Sigma)$ a scheme glued from Spec$(\mathcal{R}_\Sigma(U))$ for $U \subset TV(\Sigma)$ open.

# Application II: Connection to our work

Can apply our construction to $I$ and $C \rightsquigarrow \pi_C : \mathrm{Spec}(\mathcal{A}_C) \to \mathbb{A}^m$.

**Corollary (B–Mohammadi–Nájera Chávez)**

*Let $\Sigma$ be as above, then for every $p \in TV(\Sigma)$ there exists $a \in \mathbb{A}^m$ such that*

$$\psi_\Sigma^{-1}(p) \cong \pi_C^{-1}(a) \cong V(in_\tau(I))$$

*for some $\tau \in \Sigma \subset C \cap Trop(I)$. Moreover, if $C$ is simplicial we have a natural inclusion $TV(\Sigma) \hookrightarrow \mathbb{A}^m$, so*

$$\begin{array}{ccccc} \mathbf{Spec}(\mathcal{R}_\Sigma) & \longleftarrow & \psi_\Sigma^{-1}(p) \cong \pi_C^{-1}(\iota(p)) & \longrightarrow & \mathrm{Spec}(\mathcal{A}_C) \\ \downarrow \psi_\Sigma & & & & \downarrow \pi_C \\ TV(\Sigma) & & \overset{\iota}{\hookrightarrow} & & \mathbb{A}^m \end{array}$$

# Application II: Summary

There are two flat families

$$\psi_\Sigma : \mathbf{Spec}(\mathcal{R}_\Sigma) \to TV(\Sigma) \quad \text{and} \quad \pi_C : \mathrm{Spec}(\mathcal{A}_C) \to \mathbb{A}^m,$$

both degenerate $V(I)$ to toric varieties $V(\mathrm{in}_{\sigma_i}(I))$ that *"share a common basis"*.

- $\psi_\Sigma$ has nice geometric properties ($T$-equivariant, reduced and irreducible fibers), *but* the construction is not very explicit;
- $\pi_C$ does not have as nice geometric properties, *but* the construction is simple and well-adapted for computations.

**Thank you!**

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