# Gröbner theory of Grassmannian cluster algebras

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# Motivation

$X$ a projective variety, a *toric degeneration* of $X$ is a flat morphism $\pi : \mathfrak{X} \to \mathbb{A}^d$ with generic fibre isomorphic to $X$ and special fibre $\pi^{-1}(0)$ a toric variety.

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Question: How are different toric degenerations of $X$ related?

# Toric degenerations from valuations

$A = \bigoplus_{i \geq 0} A_i$ graded algebra and domain, $\mathfrak{v} : A \setminus \{0\} \to \mathbb{Z}^d$ a valuation with *image*$(\mathfrak{v})$ a finitely generated semigroup of rank $d := \dim_{\text{Krull}} A$.

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[Anderson] Exists a toric degeneration of $Proj(A)$ with special fibre a projective toric variety whose normalization is $TV(\Delta(A, \mathfrak{v}))$, where

$$\Delta(A, \mathfrak{v}) := \overline{\mathrm{conv}\left( \bigcup_{i \geq 1} \left\{ \frac{\mathfrak{v}(f)}{i} : f \in A_i \right\} \right)}$$

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A set $\{b_1, \ldots, b_n\} \subset A$ of algebra generators is a *Khovanskii basis* for $\mathfrak{v}$ if $\mathfrak{v}(b_1), \ldots, \mathfrak{v}(b_n)$ generate *image*$(\mathfrak{v})$.

# Gröbner toric degenerations

Reminder: $f = x^2 + y \in \mathbb{C}[x, y]$ and $w = (1, 1)$, then $in_w(f) = y$ and for $J \subset \mathbb{C}[x_1, \ldots, x_n]$ ideal $in_w(J) := (in_w(f) : f \in J)$

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Let $A := \mathbb{C}[x_1, \ldots, x_n]/J$ with $J$ homogeneous prime ideal and $w \in Trop(J)$ such that $in_w(J)$ is binomial and prime (i.e. *toric*).

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Then exists a flat family with generic fibre $Proj(A)$ and special fibre the toric variety $Proj(\mathbb{C}[x_1, \dots, x_n]/in_w(J))$, called a *Gröbner toric degeneration*.

# Motivating result

**Theorem (Kaveh–Manon, B.)**

*Let $A$ be a positively graded algebra and domain, $\mathfrak{v} : A \setminus \{0\} \to \mathbb{Z}^d$ full rank valuation with finitely generated value semigroup.*

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**Theorem (Kaveh–Manon, B.)**

*Let $A$ be a positively graded algebra and domain, $\mathfrak{v} : A \setminus \{0\} \to \mathbb{Z}^d$ full rank valuation with finitely generated value semigroup. Then there exists an isomorphism of graded algebras*

$$\mathbb{C}[x_1, \dots, x_n]/J \cong A$$

*such that the toric variety of the Newton–Okounkov polytope is isomorphic to the toric variety of a Gröbner toric degeneration for some $w \in Trop(J)$:*

$$TV(\Delta(A, \mathfrak{v})) \cong Proj(\mathbb{C}[x_1, \dots, x_n]/in_w(J))^{nor}.$$

# Motivating result

Idea of Proof: Choose a finite Khovanskii basis $b_1, \dots, b_n \in A$. Take

$$\pi : \mathbb{C}[x_1, \dots, x_n] \to A, \quad x_i \mapsto b_i$$

and $J := \ker(\pi)$.

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$$in_{w_{\mathfrak{v}}}(J) \quad \text{is toric} \quad \Leftrightarrow \quad image(\mathfrak{v}) \quad \text{is finitely generated.}$$

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Recall: $\mathfrak{v}$ defines a filtration on $A$: $F_{m;\mathfrak{v}} := \{f \in A : \mathfrak{v}(f) \leq m\}$ for all $m \in \mathbb{Z}^d$ and $\leq$ a fixed total order. A vector space basis $\mathbb{B}$ of $A$ is *adapted* to $\mathfrak{v}$ if $\mathbb{B} \cap F_{m;\mathfrak{v}}$ is a vector space basis for each $F_{m;\mathfrak{v}}$.

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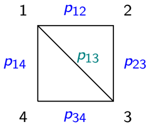


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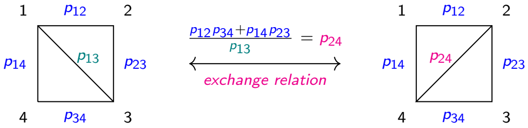


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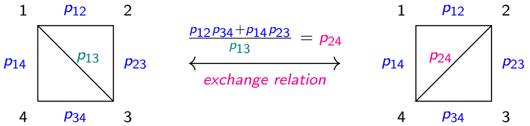


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For all seeds $s$ of $A_{k,n}$ exists a full rank valuation $g_s : A_{k,n} \setminus \{0\} \to \mathbb{Z}^d$ and a basis called $\vartheta$-basis adapted to all of them simultaneously. The *cluster algebra with principal coefficients at s* $A_{k,n}^{\text{prin},s}$ is a flat $\mathbb{C}[t_x : x \in s_{\text{mut}}]$-algebra defining the toric degeneration.

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For each seed $s$ we can apply the Motivating Theorem and get an ideal $J_s$ and a Gröbner toric degeneration of $J_s$ corresponding to $A_{k,n}^{\text{prin},s}$.

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Question: How are different $J_s$ related and what is $A_{k,n}^{\text{univ}}$ in this context?

# Gröbner fan and standard monomial bases

**Definition/Proposition (Mora–Robbiano)**

For a homogeneous ideal $J \subset \mathbb{C}[x_1, \dots, x_n]$ its *Gröbner fan* $GF(J)$ is $\mathbb{R}^n$ with fan structure defined by

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Then $\mathbb{B}_{C,\tau}$ is a vector space basis for $A_\tau$ called *standard monomial basis*.

# Example

Take $J = (x_1^2x_2^2 + x_1^4 + x_2x_3^3) \subset \mathbb{C}[x_1, x_2, x_3]$. Then $GF(J)$ is $\mathbb{R}^3$ with the fan structure:

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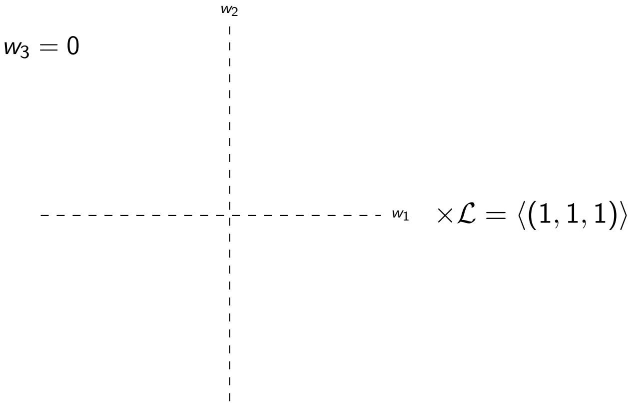

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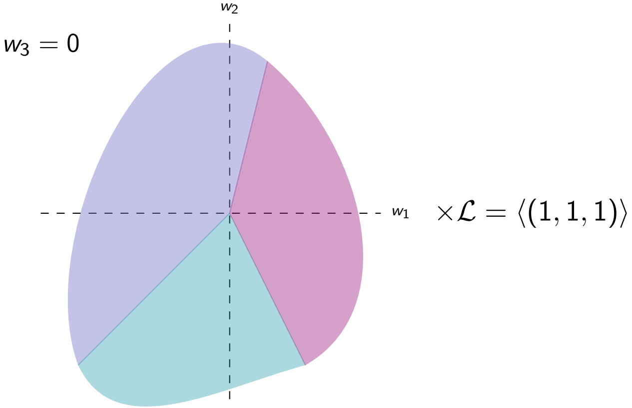

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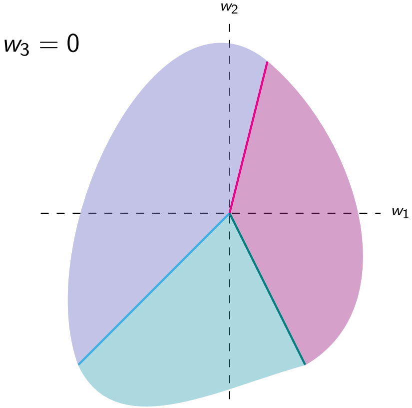


$\times \mathcal{L} = \langle (1, 1, 1) \rangle$

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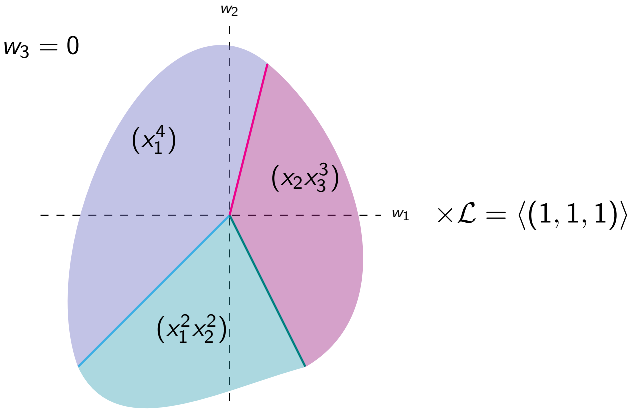

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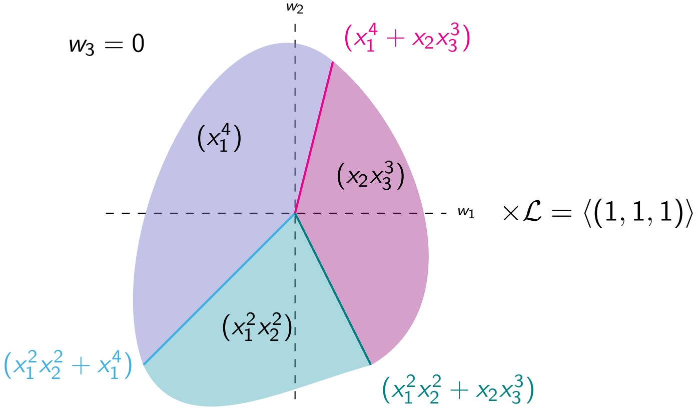

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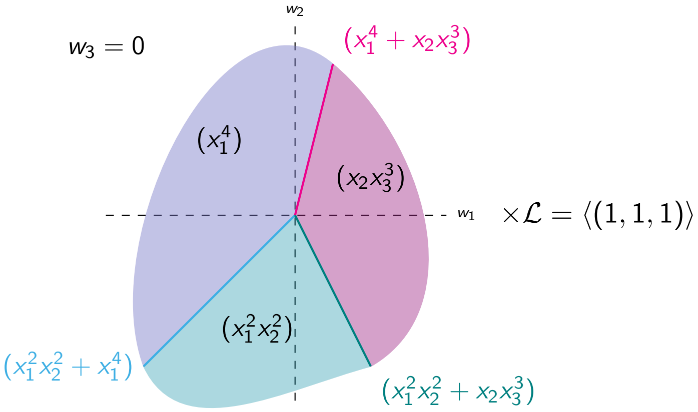


E.g. $\mathbb{B}_{\langle r_1, r_2\rangle} = \{\bar{\mathbf{x}}^a : x_2x_3^3 \nmid \mathbf{x}^a\}$ gives a basis for $A, A_{r_1}, A_{r_2}$ and $A_{\langle r_1, r_2\rangle}$.

# Family of ideals

Let $C \in GF(J)$ be a maximal cone and choose $r_1, \ldots, r_m$ representatives of primitive ray generators of $\overline{C} \in \mathrm{GF}(J)/\mathcal{L}$. Let $\mathbf{r}$ be the matrix with rows $r_1, \ldots, r_m$.

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$$\mu(f) := (\min_{c_\alpha \neq 0}\{r_1 \cdot \alpha\}, \dots, \min_{c_\alpha \neq 0}\{r_m \cdot \alpha\}) \in \mathbb{Z}^m.$$

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In $\mathbb{C}[t_1, \ldots, t_m][x_1, \ldots, x_n]$ the *lift* of $f$ is

$$\tilde{f} := f(\mathbf{t}^{\mathbf{r}\cdot e_1} x_1, \ldots, \mathbf{t}^{\mathbf{r}\cdot e_n} x_n)\mathbf{t}^{-\mu(f)} = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha \mathbf{x}^\alpha \mathbf{t}^{\mathbf{r}\cdot\alpha - \mu(f)}.$$

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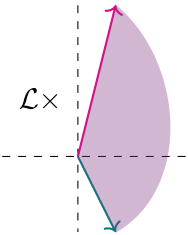
### Definition/Proposition

The *lifted ideal* $\tilde{J} := (\tilde{f} : f \in J) \subset \mathbb{C}[t_1, \dots, t_m][x_1, \dots, x_n]$ is generated by $\{\tilde{g} : g \in \mathcal{G}\}$, where $\mathcal{G}$ is a *Gröbner basis* for $J$ and $C$.

# Example

Take $f = x_1^2x_2^2 + x_1^4 + x_2x_3^3 \in \mathbb{C}[x_1, x_2, x_3]$ and consider in $GF((f))$ the maximal cone $C$ spanned by the rows of $r := \begin{pmatrix} 1 & 4 & 0 \\ 1 & -2 & 0 \end{pmatrix}$ and $\mathcal{L}$.



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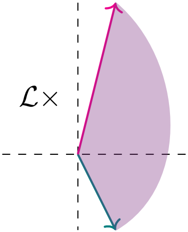
$$\tilde{f}(t_1, t_2) \quad = \quad f(t_1t_2x_1, t_1^4t_2^{-2}x_2, x_3)t_1^{-4}t_2^2$$



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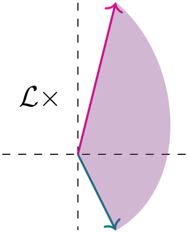

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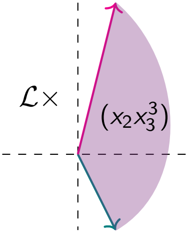


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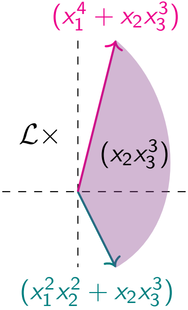

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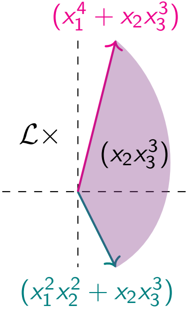

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*$\tilde{A}$ is a free $\mathbb{C}[t_1, \ldots, t_m]$-module with basis $\mathbb{B}_C$ and so the morphism*

$$\pi : Spec(\tilde{A}) \to \mathbb{A}^m$$

*is flat. In particular, $\pi$ defines a flat family with generic fiber $Spec(A)$ and for every face $\tau \subseteq C$ there exists $\mathbf{a}_\tau \in \mathbb{A}^m$ and a special fiber $\pi^{-1}(\mathbf{a}_\tau) \cong Spec(A_\tau)$.*

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Example: $\tilde{A} = \mathbb{C}[t_1, t_2][x_1, x_2, x_3]/(t_1^6 x_1^2 x_2^2 + t_2^6 x_1^4 + x_2 x_3^3)$.

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Question: How are $\tilde{A}$ and $A_{k,n}^{\mathrm{univ}}$ related?

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where $J$ is the saturation of the ideal generated by exchange relations. [Fomin–Williams–Zelevinsky, §6.8]

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The cluster variables of $A_{3,6}$ are Plücker coordinates and two more, so

$$A_{3,6} \cong \mathbb{C}[X, Y, p_{123}, \dots, p_{456}]/J_{3,6}$$

and $J_{3,6} \cap \mathbb{C}[p_{123}, \dots, p_{456}]$ is the Plücker ideal $I_{3,6}$.

A minimal generating set of $J_{3,6} \subset \mathbb{C}[p_{123}, \ldots, p_{456}, X, Y]$:

$$
\begin{array}{ll}
p_{145}p_{236} - p_{123}p_{456} - X, & p_{124}p_{356} - p_{123}p_{456} - Y, \\
p_{136}p_{245} - p_{126}p_{345} - X, & p_{125}p_{346} - p_{126}p_{345} - Y, \\
p_{146}p_{235} - p_{156}p_{234} - X, & p_{134}p_{256} - p_{156}p_{234} - Y, \\
p_{246}p_{356} - p_{346}p_{256} - p_{236}p_{456}, & p_{245}p_{356} - p_{345}p_{256} - p_{235}p_{456}, \\
p_{146}p_{356} - p_{346}p_{156} - p_{136}p_{456}, & p_{145}p_{356} - p_{345}p_{156} - p_{135}p_{456}, \\
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p_{146}p_{256} - p_{246}p_{156} - p_{126}p_{456}, & p_{145}p_{256} - p_{245}p_{156} - p_{125}p_{456}, \\
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p_{124}p_{235} - p_{234}p_{125} - p_{123}p_{245}, & p_{135}p_{146} - p_{145}p_{136} - p_{134}p_{156}, \\
p_{125}p_{146} - p_{145}p_{126} - p_{124}p_{156}, & p_{125}p_{136} - p_{135}p_{126} - p_{123}p_{156}, \\
p_{124}p_{136} - p_{134}p_{126} - p_{123}p_{146}, & p_{124}p_{135} - p_{134}p_{125} - p_{123}p_{145},
\end{array}
$$

$$f = p_{135}p_{246} - p_{156}p_{234} - Y - p_{123}p_{456} - X - p_{126}p_{345}.$$

# The totally positive part of Trop($J$)

Reminder:

$J \subset \mathbb{R}[x_1, \dots, x_n]$ is *totally positive* if $J \cap (\mathbb{R}_{>0}[x_1, \dots, x_n] \setminus \{0\}) = \varnothing$

$\overset{\text{[ET01]}}{\Leftrightarrow}$ exists $w \in \mathbb{R}^n$ such that $(\mathbb{R}_{>0})^n \cap V(in_w(J)) \neq \varnothing$.

$$Trop^+(J) := \{w \in Trop(J) : in_w(J) \text{ totally positive}\}.$$

Example:

$(p_{12}p_{34} - p_{13}p_{24})$ is totally positive, but $(p_{12}p_{34} + p_{14}p_{23})$ is not.

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3. *standard monomial basis* $\mathbb{B}_C$ = *basis of cluster monomials;*
4. $C \cap Trop(I_{k,n}) = Trop^+(I_{k,n})$ *which is a geometric realization of the cluster complex:*

$$\text{rays of } Trop^+(I_{k,n}) \quad \overset{1:1}{\longleftrightarrow} \quad \text{cluster variables}$$

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[Ilten–Nájera Chávez–Treffinger]: generalized (1) for graded cluster algebras of finte type and (2)/(3) for ADE types.

The reduced Gröbner basis of $J_{3,6}$ for $C$ consists contains the above minimal generating set and additionally the following elements:

$$\begin{aligned}
&p_{235}Y - p_{125}p_{234}p_{356} - p_{123}p_{256}p_{345}, && p_{134}X - p_{136}p_{145}p_{234} - p_{123}p_{146}p_{345},\\
&p_{146}Y - p_{124}p_{156}p_{346} - p_{126}p_{134}p_{456}, && p_{256}X - p_{156}p_{236}p_{245} - p_{126}p_{235}p_{456},\\
&p_{136}Y - p_{123}p_{156}p_{346} - p_{126}p_{134}p_{356}, && p_{346}X - p_{136}p_{234}p_{456} - p_{146}p_{236}p_{345},\\
&p_{245}Y - p_{125}p_{234}p_{456} - p_{124}p_{256}p_{345}, && p_{125}X - p_{123}p_{156}p_{245} - p_{126}p_{145}p_{235},\\
&p_{145}Y - p_{125}p_{134}p_{456} - p_{124}p_{156}p_{345}, && p_{124}X - p_{126}p_{145}p_{234} - p_{123}p_{146}p_{245},\\
&p_{236}Y - p_{126}p_{234}p_{356} - p_{123}p_{256}p_{346}, && p_{356}X - p_{136}p_{235}p_{456} - p_{156}p_{236}p_{345},\\
&p_{135}Y - p_{125}p_{134}p_{356} - p_{123}p_{156}p_{345}, && p_{135}X - p_{136}p_{145}p_{235} - p_{123}p_{156}p_{345},\\
&p_{246}Y - p_{124}p_{256}p_{346} - p_{126}p_{234}p_{456}, && p_{246}X - p_{146}p_{236}p_{245} - p_{126}p_{234}p_{456}.
\end{aligned}$$

$$g = XY - p_{123}p_{156}p_{246}p_{345} - p_{126}p_{135}p_{234}p_{456} - p_{126}p_{156}p_{234}p_{345} - p_{123}p_{156}p_{234}p_{456} - p_{123}p_{126}p_{345}p_{456}.$$

The first monomial of each relation lies in $in_C(J_{3,6})$.

# Thank you!

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# Combinatorics of $C \in GF(J_{3,6})$

Let $\{e_{123}, \ldots, e_{456}, e_x, e_y\}$ be the standard basis of $\mathbb{R}^{22}$ and

$$E_i := \sum_{k,j \neq i} e_{ijk} + e_x + e_y.$$

The *lineality space* of $GF(J_{3,6})$ is $\mathcal{L} = \langle E_1, \ldots, E_6 \rangle$. Let $f_{i,j} := \sum_{k \notin \{i,j\}} e_{ijk}$ and $\pi : \mathbb{R}^{22} \to \mathbb{R}^{20}$ away from $e_x, e_y$.

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| # | rays of $C/\mathcal{L}$ | projections |
|---|---|---|
| 6 | $a_i := e_{i,i+1,i+2}$ | $e_{i,i+1,i+2}$ |
| 6 | $b_i := f_{i,i+1} + \delta_{i \text{ odd}} e_y + \delta_{i \text{ even}} e_x$ | $f_{i,i+1}$ |
| 2 | $c_i^- := b_i + e_{i-2,i-1,i} + e_{i-2,i-1,i+1}$ | $g_{i,i+1,i+2,i-3,i-2,i-1}$ |
| 2 | $c_i^+ := b_i + e_{i,i+2,i+3} + e_{i+1,i+2,i+3}$ | $g_{i+2,i+1,i,i-3,i-2,i-1}$ |

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Notice: $c_i^{\pm} = c_j^{\pm} mod \mathcal{L}$ if $i = j \mod 2$ and $g_{i,i+1,i+2,i-3,i-2,i-1} + g_{i+2,i+1,i,i-3,i-2,i-1} = f_{i+1,i+2} + f_{i-1,i} + f_{i-2,i-3}$

# Combinatorics of $C \in GF(J_{3,6})$

The ideal $J_{3,6}$ is invariant under the action of $G := \langle (123456), w_0 \rangle \subset \mathfrak{S}_6$, so $GF(J_{3,6})$ has an induced *G-action*.

[1]FFFGG is a bipyramid in $\mathrm{Trop}(I_{3,6})$ and each $G$-orbit maps onto one of the pyramids.

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| # | $G$-orbit of max cone in $C \cap Trop(J_{3,6})$ | type of projection | # |
|---|---|---|---|
| 18 | $\{a_i, a_{i-2}, b_{i-2}, b_{i+3}\}$ | EEFF | 180 |
| 12 | $\{a_i, c_i^{\pm}, b_{i-2}, b_{i+4}\}$ | EFFG | 180 |
| 12 | $\{a_i, a_{i+2}, b_i, c_i^{+}\}$ or $\{a_i, a_{i-2}, b_{i+1}, c_{i+1}^{-}\}$ | EEFG | 360 |
| 4 | $\{a_{i-2}, a_i, a_{i+2}, c_i^{\pm}\}$ | EEEG | 240 |
| 4 | $\{b_{i-2}, b_i, b_{i+2}, c_i^{\pm}\}$ | FFFGG[1] | 15 |

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| 4 | $\{b_{i-2}, b_i, b_{i+2}, c_i^{\pm}\}$ | FFFGG[1] | 15 |

The type of projected cone refers to the $\mathfrak{S}_6$-orbits in Trop$(I_{3,6})$, respectively Trop$^+(I_{3,6})$, as used [SS04]&[BCL17], the number is the number of maximal cones in Trop$(I_{3,6})$ of this type.

[1] FFFGG is a bipyramid in Trop$(I_{3,6})$ and each $G$-orbit maps onto one of the pyramids.