Newton–Okounkov bodies for cluster varieties

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Motivation

Let $X = G/P \hookrightarrow \mathbb{P}(V_{\lambda})$ a flag variety and $R = \bigoplus_{\ell \geq 0} V_{\ell \lambda}^*$ its homogeneous coordinate ring.

Given a valuation $\nu: R \to \mathbb{Z}$ of $\text{dim } X + 1$ with finitely generated image of rank $\text{dim } X + 1$ and its Newton–Okounkov polytope $\Delta(\nu, R) := \text{conv} \bigcup_{i \geq 1} \{ \nu(f)_{\ell} : f \in V_{\ell \lambda}^* \}$ we get a toric degeneration of $X$ to $X_{\Delta}$.

Many polytopes parametrizing bases of representations arise this way, like Gelfand–Zetlin polytopes, string polytopes, FFLV polytopes, ... Further, many of the mentioned polytopes are isomorphic to polytopes arising from tropicalized potentials on cluster varieties contained in $X$.

Aim: Develop the framework of Newton–Okounkov bodies for cluster varieties that includes all the representation theoretic examples.
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Overview

1. Cluster varieties
   - Tropicalization
   - Fock–Goncharov conjecture
   - Wall and chamber structure

2. Compactifications
   - Potentials

3. Intrinsic Newton–Okounkov bodies
   - Broken line convexity

4. Grassmannians
Part I: Cluster varieties

$N \cong \mathbb{Z}^n$ lattice, $\{\cdot, \cdot\} : N \times N \rightarrow \mathbb{Z}$ skew-symmetric bilinear form, $M = N^*$

$$\mu(n,m) : T_N := N \otimes_{\mathbb{Z}} \mathbb{C}^* \rightarrow T_N \quad \text{called mutation}$$

$$\mu^*(n,m)(z^{m'}) = z^{m'}(1 + z^m)^{m'(n)}.$$

Exercise:

Tropicalization of $\mu(-e_k, v_k)$ to $\mu_T k$:

$X := \bigcup_{s \sim s_0} T_M$, $s$ glued by mutations $\mu(v_k, e_k)$

$\Rightarrow$ dual cluster varieties $A$ and $X$ generalize dual tori $T_N$ and $T_M$. 

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Let \( s_0 = \{e_1, \ldots, e_n\} \) basis of \( N \) (called a seed) and \( \nu_i := \{e_i, \cdot\} \in M \)

**Exercise:** Tropicalization of \( \nu(-e_k, \nu_k) \) to \( \nu_k^T : T_N(\mathbb{Z}^T) = N \to N \) is a pseudoreflection and \( \nu_k^T(s_0) \) is a new seed.
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\textit{A-cluster varieties} are the schemes

$$
\mathcal{A} := \bigcup_{s \sim s_0} T_{N,s} \quad \text{glued by mutations } \mu(-e_k, v_k)
$$

$$
\mathcal{X} := \bigcup_{s \sim s_0} T_{M,s} \quad \text{glued by mutations } \mu(v_k, e_k)
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\textit{\sim} \textit{ dual cluster varieties } \mathcal{A} \text{ and } \mathcal{X} \text{ generalize dual tori } T_N \text{ and } T_M.
Example: $\mathcal{A}$ and $\mathcal{X}$ in case $A_2$

$N = \mathbb{Z}^2$ with $\{\cdot, \cdot\}$ given by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for $\{e_1, e_2\}$. Then $\mathcal{A}$ and $\mathcal{X}$ are glued from 5 tori each with local coordinates:
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Example: cluster variety inside $\text{Gr}_{2,n}$

Let $N = \mathbb{Z}^{2(n-2)+1}$ with seed basis $\{e_{12}, \ldots, e_{1n}, e_{23}, e_{34}, \ldots, e_{n-1,n}\}$ and all $e_{i,i+1}$ frozen. The form $\{\cdot, \cdot\}$ is given by

$$\begin{array}{cccc}
& & & \\
& p_{12} & & p_{15} \\
& & p_{13} & & \\
& p_{23} & & p_{14} & \\
& & & p_{34} & \\
& & & & p_{45}
\end{array}$$

If we identify $z^{f_{ij}} = p_{ij}$ then $\mathcal{A} \subset \widehat{\text{Gr}_2}(\mathbb{C}^n)$. More precisely,

$$\mathcal{A} = \widehat{\text{Gr}_2}(\mathbb{C}^n) \setminus \bigcup_{i=1}^{n-1} \{p_{i,i+1} = 0\}.$$
Example: cluster variety inside $\text{Gr}_{2,5}$

In this case, we have a bijection between

seeds $\leftrightarrow$ triangulations of an $n$-gon

The cluster variables $z^f_{ij}$ are Plücker coordinates and the pull-back of the $\mathcal{A}$-cluster mutation on those corresponds to three-term Plücker relations.
Tropicalizing cluster varieties

Notice: Mutation $\mu^*_{(n,m)}(z^{m'}) = z^{m'}(1 + z^m)^{m'(n)}$ is substraction-free

⇒ may consider cluster varieties over semifields.

For $\mathbb{P}$ a semifield we have $T_N(\mathbb{P}) = N \otimes \mathbb{Z} \mathbb{P}$.
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⇒ may consider cluster varieties over *semifields*.

For $\mathbb{P}$ a semifield we have $T_N(\mathbb{P}) = N \otimes \mathbb{P}$.

⇒ every seed $s$ gives *non-canonical* identifications

$$A(\mathbb{P})|_{T_{N,s}} \equiv N \otimes \mathbb{P} \quad \text{and} \quad \mathcal{X}(\mathbb{P})|_{T_{M,s}} \equiv M \otimes \mathbb{P}$$
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\]

Definition

The *(integer/rational/real) tropicalization* of a cluster variety is

\[
\mathcal{A}(\mathbb{Z}^T)/\mathcal{A}(\mathbb{Q}^T)/\mathcal{A}(\mathbb{R}^T) \quad \text{resp.} \quad \mathcal{X}(\mathbb{Z}^T)/\mathcal{X}(\mathbb{Q}^T)/\mathcal{X}(\mathbb{R}^T),
\]

where \( \mathbb{Z}^T = (\mathbb{Z}, \text{max}, +)/\mathbb{Q}^T = (\mathbb{Q}, \text{max}, +)/\mathbb{R}^T = (\mathbb{R}, \text{max}, +) \).
Cluster duality and the Fock–Goncharov conjecture

Recall: $T_N$ has dual torus $T_M$ and $T_M(\mathbb{Z}^T) = M \otimes \mathbb{Z} \mathbb{Z} = M$ parametrizes a basis of regular functions $\Gamma(T_N, \mathcal{O}_{T_N}) \leadsto \text{characters of } T_N$
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Fock–Goncharov conjecture

The tropical cluster variety $\mathcal{X}(\mathbb{Z}^T)$, respectively $\mathcal{A}(\mathbb{Z}^T)$, parametrizes a basis for $\Gamma(\mathcal{A}, \mathcal{O}_A)$, respectively $\Gamma(\mathcal{X}, \mathcal{O}_X)$. 
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false in general (counter examples due to Gross–Hacking–Keel),

true in interesting examples like cluster varieties inside the Grassmannians, flag varieties, configuration space.

Assumption: the full Fock–Goncharov conjecture holds for $\mathcal{A}$, that is $\Theta := \{ \vartheta_m : m \in \mathcal{X}(\mathbb{Z}^T) \}$ is a basis for $\Gamma(\mathcal{A}, \mathcal{O}_\mathcal{A})$, called theta basis.
Wall and chamber structure on $\mathcal{X}(\mathbb{R}^T)$

**Fact:** for every seed $s' = (e'_1, \ldots, e'_n)$ with dual basis $f'_1, \ldots, f'_n \in M$ we have $z^{m_1 f'_1 + \cdots + m_n f'_n} \in \Theta$ with $m_i \in \mathbb{N}$ called *cluster monomials* and

$$g_{s'}(z^{m_1 f'_1 + \cdots + m_n f'_n}) = m_1 f'_1 + \cdots + m_n f'_n$$

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Let $G_{s_0}(s') = \mu^*_{s_0, s'}(\langle f'_1, \ldots, f'_n \rangle \geq 0)$ then $\bigcup_{s' \sim s_0} G_{s_0}(s')$ is a simplicial fan\(^1\)

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**Example:** In case of $\mathcal{A} \subset \text{Gr}_2(\mathbb{C}^5)$, so $N = \mathbb{Z}^{\binom{5}{2}}$, consider a slice of $M$ of points $af_{13} + bf_{14}$, $a, b \in \mathbb{Z}$:

$$\vartheta(-a,a+b) = p_{24}^a p_{14}^b$$

$$\vartheta(a,b) = z^{af_1+bf_2} = p_{13}^a p_{14}^b$$

$$p_{24} = \frac{p_{12}p_{34}^2 + p_{14}p_{23}}{p_{13}} = \vartheta(-1,1)$$

$$\vartheta(0,1) = z^{f_2} = p_{14}$$

$$\vartheta(1,0) = z^{f_1} = p_{13}$$

$M_{\mathbb{R}} \equiv \mathcal{X}(\mathbb{R}^T)$

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1 conjectured by [FZ], partial results due to [CIKLP], full generality [GHKK]
In the initial data $s_0 = \{e_1, \ldots, e_n\} \subset N$ declare $e_k, \ldots, e_n$ frozen, i.e. never mutate at $e_k, \ldots, e_n$, then allow vanishing of $z_{f_k}, \ldots, z_{f_n}$. 

Example: $N = \mathbb{Z}_2$ with $\{·, ·\}$ given by

\[
\begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}
\]

for $\{e_1, e_2\}$ and $e_2$ frozen. Then $A$ is glued from 2 tori each with local coordinates:

\[
\sigma_0 \sigma_4 z_{f_1} z_{f_2} z_{f_2}^1 + z_{f_2} z_{f_1}
\]

The (partial) compactification $A$ is glued from two copies of $\mathbb{C}^* \times \mathbb{C}$ along the biggest open subset where mutation is still defined. $D := A \setminus A$ is called the boundary divisor.
Part II: Compactifying \( \mathcal{A} \)-cluster varieties

In the initial data \( s_0 = \{e_1, \ldots, e_n\} \subset N \) declare \( e_k, \ldots, e_n \) frozen, i.e. never mutate at \( e_k, \ldots, e_n \), then allow vanishing of \( z^{f_k}, \ldots, z^{f_n} \).

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The (partial) compactification $\overline{\mathcal{A}}$ is glued from two copies of $\mathbb{C}^* \times \mathbb{C}$ along the biggest open subset where mutation is still defined.

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Potentials for cluster varieties

In the spirit of mirror symmetry the compactification $\overline{A}$ of $A$ should induce a potential function on the dual $\mathcal{X}$ cluster variety.

Recall: $D = D_k \cup \cdots \cup D_n$ with $D_i = \{z^{f_i} = 0\}$. 

Note: $\sigma_0 \sigma_4 = \{m \in M_D : \langle m, -e_1 \rangle \leq 0, \langle m, -e_1 - e_2 \rangle \leq 0\}$. 

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Mild assumptions $\Rightarrow$ may identify tropical points with divisorial discrete valuations:

$$\text{ord}_{D_i} \longleftrightarrow n_i \in A(\mathbb{Z}^T) \longleftrightarrow \vartheta_i : X \to \mathbb{C}$$

Then the $\vartheta$-potential is $W = \vartheta_k + \cdots + \vartheta_n : X \to \mathbb{C}$. 
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Combinatorial hypothesis $\Rightarrow$ for every $i$ there exists a seed $s' = (e_1', \ldots, e_n')$ and $\vartheta_i|_{T_M, s'} = z^{-e_i'}$.  

Example: $N = \mathbb{Z}_2$ with $\{\cdot, \cdot\}$ given by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for $s_0 = \{e_1, e_2\}$ and $e_2$ frozen we have $D = \{z^{f_2} = 0\}$ and $W|_{T_M, s_0} = \vartheta_1|_{T_M, s_0} = z - e_2 + z - e_1 - e_2$. 

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Valuations for cluster varieties

Recall: $X(Z^T)_{M,s} \equiv M$ and the theta basis $\Theta = \{ \vartheta_m : m \in X(Z^T) \}$.

Be $(\bar{\mathcal{A}}, D)$ a (partially) compactified cluster variety with theta potential $W : \mathcal{X} \to \mathbb{C}$ and its tropicalization $\Xi := \{ m \in X(Z^T) : W^T(m) \leq 0 \}$. 

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Valuations for cluster varieties

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$$\overline{\Theta} := \{\vartheta_m : m \in \Xi\} \text{ is a basis for } \Gamma(\overline{A}, O_{\overline{A}}).$$

Let $\Xi_s := \text{Cone}(\Xi \cap \mathcal{X}(\mathbb{Z}^T)|_{T_{M,s}}) \subset M_{\mathbb{R}}$. 
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Proposition (B.–Cheung–Magee–Nájera Chávez)

Given the above assumptions the assignment \( \vartheta_m \mapsto m \in \Xi_s \) for \( m \in \Xi_s \cap M \) extends to a valuation

\[ g_s : \Gamma(\overline{A}, O_{\overline{A}}) \setminus \{ 0 \} \to M \]

whose Newton–Okounkov cone is \( \Xi_s \).
Part III: Broken line convexity

\[ \mathcal{X}(\mathbb{Z}^T)|_{T_{M,s}} \equiv M \] is non-canonical as \( \mathcal{X}(\mathbb{Z}^T) \) is not a lattice.

But \( \mathcal{X}(\mathbb{R}^T) \) has a wall and chamber structure and notion of broken lines\(^2\).

\[^2\text{combinatorial gadgets replacing pseudoholomorphic disks, their counts give the structure constants for the theta basis}\]
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But \( \mathcal{X}(\mathbb{R}^T) \) has a **wall and chamber structure** and notion of **broken lines**\(^2\).

[Cheung–Magee–Nájera Chávez] introduce **broken line convexity**: a closed set \( S \subset \mathcal{X}(\mathbb{R}^T) \) is broken line convex iff \( \forall a, b \in S \) and any broken line segment \( \ell \) between \( a, b \) we have \( \ell \subset S \).

\(^2\)combinatorial gadgets replacing pseudoholomorphomorphic disks, their counts give the structure constants for the theta basis.
Part III: Broken line convexity

\[ \mathcal{X}(\mathbb{Z}^T)|_{T_{M,s}} \equiv M \text{ is non-canonical as } \mathcal{X}(\mathbb{Z}^T) \text{ is not a lattice.} \]

But \( \mathcal{X}(\mathbb{R}^T) \) has a \textit{wall and chamber structure} and notion of \textit{broken lines}\(^2\).

[Cheung–Magee–Nájera Chávez] introduce \textit{broken line convexity}: a closed set \( S \subset \mathcal{X}(\mathbb{R}^T) \) is broken line convex iff \( \forall a, b \in S \) and any broken line segment \( \ell \) between \( a, b \) we have \( \ell \subset S \).

\[ \text{Theorem (Cheung–Magee–Nájera Chávez)} \]

A compact set \( S \subset \mathcal{X}(\mathbb{R}^T) \) that is broken line convex defines a projective compactification of an \( \mathcal{A} \)-cluster variety whose graded ring has a theta basis.

\(^2\)Combinatorial gadgets replacing pseudoholomorphic disks, their counts give the structure constants for the theta basis.
Intrinsic Newton–Okounkov body

Assuming the full Fock–Goncharov conjecture holds, for \( f \in \Gamma(A, \mathcal{O}_A) \) we have \( f = \sum_{m \in \mathcal{X}(\mathbb{Z}^T)} c_m \vartheta_m \) and define its \( \vartheta \)-Newton polytope:

\[
\text{New}_{\vartheta}(f) := \text{conv}_{BL} \left( m \in \mathcal{X}(\mathbb{Z}^T) : c_m \neq 0 \right) \subset \mathcal{X}(\mathbb{R}^T).
\]
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\]

For \( \mathcal{L} \) and line bundle on \( \overline{A} \) and \( R(\mathcal{L}) = \bigoplus_{j \geq 0} R_j(\mathcal{L}) \) its section ring we define the \emph{intrinsic Newton–Okounkov body}

\[
\Delta_{BL}(\mathcal{L}) := \text{conv}_{BL} \left( \bigcup_{j \geq 1} \left( \bigcup_{f \in R_j(\mathcal{L})} \left\{ \frac{1}{j} \text{New}_{\vartheta}(f) \right\} \right) \right) \subset X(\mathbb{R}^T).
\]
Intrinsic Newton–Okounkov body

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\]

Theorem (B.–Cheung–Magee–Nájera Chávez)

For a seed \( s \) and \( g_s : R(\mathcal{L}) \setminus \{0\} \to M \equiv \mathcal{X}(\mathbb{Z}^T)|_{T_{M,s}} \) we have

\[
\Delta_{BL}(\mathcal{L})|_{T_{M,s}} = \Delta(g_s, \mathcal{L}) = \Xi_s \cap H_\mathcal{L} \subset M_\mathbb{R}
\]

\( \rightsquigarrow \) the broken line convex hull detects missing vertices.
Example: Grassmannian

For a general Grassmannian $\text{Gr}_k(\mathbb{C}^n)$ we define the initial seed $s$ with basis $\{e_{ij}\}_{i,j}$ where $z^{f_{ij}} := p[1,k-j] \cup [k-j+i+1,k+i]$ and $\{\cdot, \cdot\}$ given by: 

\[
\text{Exercise: for } k = 2 \quad \text{Q corresponds to } \\
\begin{array}{c}
\begin{array}{c}
1 \\
2 \\
\vdots \\
\end{array} \\
\begin{array}{c}
n-1 \\
n \\
\vdots \\
\end{array}
\end{array}
\]

Mutation at 4-valent vertices $\leftrightarrow$ 3-term Plücker relations, e.g. 

\[
\mu_{1 \times 1}(z_{f_{1 \times 1}}) = z_{f_{1 \times 2}} + z_{2 \times 1} + z_{\emptyset} + z_{2 \times 2} - z_{1 \times 1}
\]

\[
\mu_{1 \times k-2}(p_{1,k-2}) \cup p_{k,k+1} + p_{1,k-1} \cup \{k+1\} = p_{1,k-2} \cup \{k,k+2\}
\]
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\[
\emptyset \\
1 \times 1 \rightarrow 1 \times 2 \rightarrow \cdots \rightarrow 1 \times k \\
\downarrow \quad \downarrow \quad \downarrow \quad \cdots \quad \downarrow \\
2 \times 1 \rightarrow 2 \times 2 \rightarrow \cdots \rightarrow 2 \times k \\
\downarrow \quad \downarrow \quad \downarrow \quad \cdots \quad \downarrow \\
\vdots \quad \vdots \quad \vdots \quad \cdots \quad \vdots \\
(n-k) \times 1 \rightarrow (n-k) \times 2 \rightarrow \cdots \rightarrow (n-k) \times k \\
\downarrow \quad \downarrow \quad \downarrow \quad \cdots \quad \downarrow \\
\vdots \quad \vdots \quad \vdots \quad \cdots \quad \vdots 
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\downarrow & & & \\
2 \times 1 & \rightarrow 2 \times 2 & \cdots & \rightarrow 2 \times k \\
\downarrow & & & \\
\vdots & & & \\
(n-k) \times 1 & \rightarrow (n-k) \times 2 & \cdots & \rightarrow (n-k) \times k
\end{align*}
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$Q$ corresponds to

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\begin{array}{c}
1 \\
\downarrow \\
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\end{array}
\]
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\downarrow & & & \\
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\downarrow & & & \\
\vdots & & & \\
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\vdots & \vdots & \ddots & \vdots \\
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\[
\begin{array}{c}
1 \\
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3 \\
n-1
\end{array}
\]

Mutation at 4-valent vertices \( \leftrightarrow \) 3-term Plücker relations, e.g.

\[
\mu_{1 \times 1}(z^{f_{1 \times 1}}) = \frac{z^{f_{1 \times 2} + f_{2 \times 1} + f_{\emptyset} + f_{\emptyset} + f_{2 \times 2}}}{z^{f_{1 \times 1}}} = \frac{p_{[1,k-2] \cup [k,k+1]} p_{[1,k-2] \cup [k+1,k+2]} + p_{[1,k-1] \cup \{k+1\}} p_{[1,k]}}{p_{[1,k-1] \cup \{k+1\}}} = p_{[1,k-2] \cup \{k,k+2\}}.
\]
The Grassmannian $\text{Gr}_{k,n}$ is a compactification of a cluster variety [Scott]. More precisely, there exists $(\overline{A}, D)$ satisfying all above conjectures and assumptions with $\Gamma(\overline{A}, O_{\overline{A}}) \cong \text{Cox}(\text{Gr}_{k,n})$. 

The cluster dual $X^\vee$ can be embedded into the dual Grassmannian $\text{Gr}_{n-k,n}$. [Marsh–Rietsch] prove a mirror symmetry conjecture for $\text{Gr}_{k,n}$ using their Plücker potential $P_{\circlearrowleft} : \text{Gr}_{n-k,n} \to \mathbb{C}$. [Rietsch–Williams] use the cluster structure to define full rank valuations $\text{val}_s : \text{Cox}(\text{Gr}_{k,n}) \{0\} \to \mathbb{N}$ and Newton–Okounkov polytopes $\Delta(\text{val}_s)$ for every seed $s$. They show $\Delta(\text{val}_s) \cong \Gamma_s(P)$ where $\Gamma_s(P) \subset \mathbb{N} \otimes \mathbb{Z}$ is the tropicalization of $P|_{\mathbb{T}\mathbb{N}_s}$.
Part IV: Examples

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Part IV: Examples

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$$
\Delta(\text{val}_s) \cong \Gamma_s(P)
$$

where $\Gamma_s(P) \subset \mathbb{N} \otimes_{\mathbb{Z}} \mathbb{R}$ is the tropicalization of $P|_{T_{N,s}}$. 
Example: $\text{Gr}_2(\mathbb{C}^5)$

In the initial chart $P : \text{Gr}^o_{3,5} \to \mathbb{C}$ is given by

$$P = \frac{p_{235}}{p_{234}} + \frac{p_{134}}{p_{345}} + \frac{p_{245}}{p_{145}} + \frac{p_{135}}{p_{125}} + \frac{p_{124}}{p_{123}}.$$

In this case,

$$\text{conv}(\{\text{val}_s(p_{ij})\}_{i,j}) = \Delta(\text{val}_s) = \Gamma_s(P)$$

is the Gelfand–Zetlin polytope for $SL_5$ and weight $\omega_2$.

| Table 2. The valuations $\text{val}_G(P_J)$ of the Plücker coordinates. |
|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|-------------------|
| Plücker          | $\square$ | $\square$ | $\square$ | $\square$ | $\square$ | $\square$ | $\square$ | $\square$ | $\square$ | $\square$ | $\square$ | $\square$ | $\square$ | $\square$ | $\square$ | $\square$ |
| $P_{1.2}$        | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
| $P_{1.3}$        | 1         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
| $P_{1.4}$        | 1         | 1         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
| $P_{1.5}$        | 1         | 1         | 1         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
| $P_{2.3}$        | 1         | 0         | 0         | 1         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
| $P_{2.4}$        | 1         | 1         | 0         | 1         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
| $P_{2.5}$        | 1         | 1         | 1         | 1         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
| $P_{3.4}$        | 2         | 1         | 0         | 1         | 1         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
| $P_{3.5}$        | 2         | 1         | 1         | 1         | 1         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
| $P_{4.5}$        | 2         | 2         | 1         | 1         | 1         | 1         | 1         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         | 0         |
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In this case,

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is the **Gelfand–Zetlin polytope** for $SL_5$

and weight $\omega_2$.

Recall, $N = \mathbb{Z}^7$ with basis $\{e_{12}, e_{13}, e_{14}, e_{15}, e_{23}, e_{34}, e_{45}\}$ and $z^{f_{ij}} = p_{ij}.$

For the same initial seed we have $W : \tilde{\mathcal{X}} \to \mathbb{C}$ given by

$$z^{-e_{15}} + z^{-e_{23}} + z^{-e_{12}}(1 + z^{-e_{13}}) + z^{-e_{45}}(1 + z^{-e_{14}}) + z^{-e_{34}}(1 + z^{-e_{13}}(1 + z^{-e_{14}})).$$

And $\Xi_s \cap (f_{12} + f_{13} + f_{14} + f_{15} + f_{23} + f_{34} + f_{45})_{\perp} \cong \Gamma_s(P).$
Grassmannians

Remark: The equality $\text{conv}(\{\text{val}_s(p_{ij})\}_{i,j}) = \Delta(\text{val}_s)$ is false in general (counterexamples for $\text{Gr}_{3,6}$ and larger). In general it is a hard (nonetheless desirable) to find vertices of a Newton–Okounkov polytope, but broken line convexity helps to do exactly this.
Remark: The equality $\text{conv}(\{\text{val}_s(p_{ij})\}_{i,j}) = \Delta(\text{val}_s)$ is *false* in general (counterexamples for $\text{Gr}_{3,6}$ and larger). In general it is a hard (nonetheless desirable) to find vertices of a Newton–Okounkov polytope, but broken line convexity helps to do exactly this.

Fact: For all $\mathcal{A}$- and $\mathcal{X}$-cluster varieties there exists a family of *cluster ensemble maps* $p : \mathcal{A} \to \mathcal{X}$ given by $p^*_s : N \to M$ for every seed $s$.

$$
\tilde{\text{Gr}}_{k,n} \supset \tilde{\mathcal{A}} \xrightarrow{\tilde{p}} \tilde{\mathcal{X}} \subset \text{Gr}_{k,n} \\
\text{Gr}_{k,n} \supset \mathcal{A} \xrightarrow{p} \mathcal{X} \subset \text{Gr}_{k,n}
$$

$$
\text{Gr}_{n-k,n} \supset \mathcal{A}^\vee \xrightarrow{p^\vee} \mathcal{X}^\vee \subset \text{Gr}_{n-k,n} \\
\tilde{\text{Gr}}_{n-k,n} \supset \tilde{\mathcal{A}}^\vee \xrightarrow{\tilde{p}^\vee} \tilde{\mathcal{X}}^\vee \subset \tilde{\text{Gr}}_{n-k,n}
$$

We have theta potentials $W : \tilde{\mathcal{X}} \to \mathbb{C}$ and $W^\vee : \tilde{\mathcal{X}}^\vee \to \mathbb{C}$ and Plücker potentials $P : \mathcal{A}^\vee \to \mathbb{C}$ and $P^\vee : \mathcal{A} \to \mathbb{C}$. 
Grassmannians

Theorem (B.–Cheung–Magee–Nájera Chávez)

For $\tilde{A} \subset \tilde{\text{Gr}}_{k,n}$ there exists a unique \textit{cluster ensemble map} $p : \tilde{A} \to \tilde{X}$ that pulls back the theta to the Plücker potential $p^*(W) = \tilde{P}^\vee$.
Grassmannians

Theorem (B.–Cheung–Magee–Nájera Chávez)

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Moreover, the dual map $(p^*)_s : N \to M$ satisfies for Plücker coordinates $p_I$

$$(p^*)_s(\nu_s(p_I)) = g_s(p_I) + c,$$

Corollary: $(p^*)_s(\Gamma_s) = \Xi_s \cap H = \Delta(g_s) = (p^*)_s(\Delta(\nu_s))$, so $\Gamma_s = \Delta(\nu_s)$. 

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Grassmannians

**Theorem (B.–Cheung–Magee–Nájera Chávez)**

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Moreover, the dual map $(p^*)^\vee_s : N \to M$ satisfies for Plücker coordinates $p_I$

$$(p^*)^\vee_s(\nu_s(p_I)) = g_s(p_I) + c,$$

and for all seeds $s$ the Newton–Okounkov body $\Delta(g_s)$ is

$$\Delta(g_s) = \text{conv}_{BL} \left( g_s(p_I) : I \in \binom{[n]}{k} \right) \big|_{T_{M,s}} \subset M_\mathbb{R}. $$

**Corollary:** $(p^*)^\vee_s(\Gamma_s) = \Xi_s \cap H = \Delta(g_s) = (p^*)^\vee_s(\Delta(\nu_s))$, so $\Gamma_s = \Delta(\nu_s)$. 
References


