Newton–Okounkov bodies for cluster varieties

Lara Bossinger (jt. Cheung, Magee, Nájera Chávez)

Universidad Nacional Autónoma de México, IM-Oaxaca

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Motivation

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$$\Delta(\nu, R) := \overline{\mathrm{conv} \bigcup_{i \geq 1} \left\{ \frac{\nu(f)}{\ell} : f \in V_{\ell\lambda}^* \right\}}$$

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Aim: Develop the framework of Newton–Okounkov bodies for cluster varieties that includes all the representation theoretic examples.

Overview

1. Cluster varieties
   - Tropicalization
   - Fock–Goncharov conjecture
   - Wall and chamber structure
2. Compactifications
   - Potentials
3. Intrinsic Newton–Okounkov bodies
   - Broken line convexity
4. Grassmannians

Part I: Cluster varieties

$N \cong \mathbb{Z}^n$ lattice, $\{\cdot,\cdot\} : N \times N \to \mathbb{Z}$ skew-symmetric bilinear form, $M = N^*$

$$\begin{aligned} \mu_{(n,m)} : T_N := N \otimes_{\mathbb{Z}} \mathbb{C}^* &\dashrightarrow T_N \qquad \text{called } \textit{mutation} \\ \mu^*_{(n,m)}(z^{m'}) &= z^{m'}(1+z^m)^{m'(n)}. \end{aligned}$$

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Let $s_0 = \{e_1, \dots, e_n\}$ basis of N (called a *seed*) and $v_i := \{e_i, \cdot\} \in M$

Exercise: Tropicalization of $\mu_{(-e_k, v_k)}$ to $\mu_k^{\mathrm{T}} : T_N(\mathbb{Z}^T) = N \to N$ is a pseudoreflection and $\mu_k^T(s_0)$ is a new seed.

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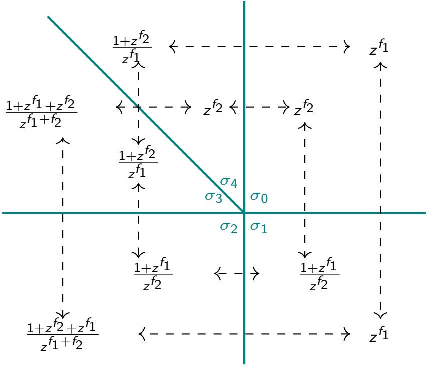
$\mathcal{A}$-cluster varieties are the schemes

$$\begin{array}{rcll} \mathcal{A} & := & \bigcup_{s\sim s_0} T_{N,s} & \text{glued by mutations } \mu_{(-e_k,v_k)} \\ \mathcal{X} & := & \bigcup_{s\sim s_0} T_{M,s} & \text{glued by mutations } \mu_{(v_k,e_k)} \end{array}$$

$\rightsquigarrow$ *dual cluster varieties $\mathcal{A}$ and $\mathcal{X}$ generalize dual tori T_N and T_M.*

Example: $\mathcal{A}$ and $\mathcal{X}$ in case $\mathtt{A}_2$

$N = \mathbb{Z}^2$ with $\{\cdot,\cdot\}$ given by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ for $\{e_1, e_2\}$. Then $\mathcal{A}$ and $\mathcal{X}$ are glued from 5 tori each with local coordinates:



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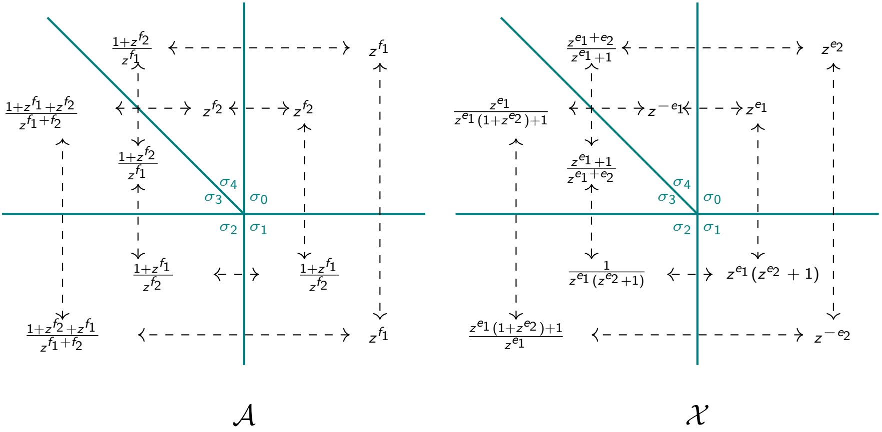
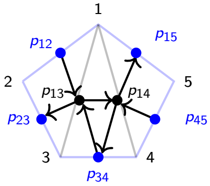


Example: cluster variety inside $\mathrm{Gr}_{2,n}$

Let $N = \mathbb{Z}^{2(n-2)+1}$ with seed basis $\{e_{12}, \dots, e_{1n}, e_{23}, e_{34}, \dots, e_{n-1,n}\}$ and all $e_{i,i+1}$ *frozen*. The form $\{\cdot,\cdot\}$ is given by



If we identify $z^{f_{ij}} = p_{ij}$ then $\mathcal{A} \subset \widetilde{\mathrm{Gr}_2(\mathbb{C}^n)}$. More precisely,

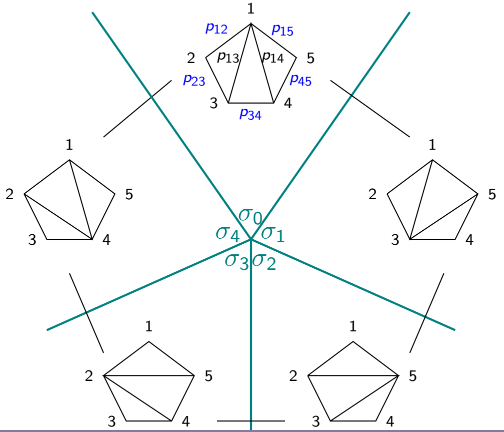
$$\mathcal{A} = \widetilde{\mathrm{Gr}_2(\mathbb{C}^n)} \setminus \bigcup_{i=1}^{n-1} \{p_{i,i+1} = 0\}.$$

Example: cluster variety inside $\mathrm{Gr}_{2,5}$

In this case, we have a bijection between

$$\text{seeds} \leftrightarrow \text{triangulations of an } n\text{-gon}$$

The cluster variables $z^{f_{ij}}$ are Plücker coordinates and the pull-back of the $\mathcal{A}$-cluster mutation on those corresponds to three-term Plücker relations.



Tropicalizing cluster varieties

Notice: Mutation $\mu^*_{(n,m)}(z^{m'}) = z^{m'}(1+z^m)^{m'(n)}$ is substraction-free
$\Rightarrow$ may consider cluster varieties over *semifields*.

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Definition

The *(integer/rational/real) tropicalization* of a cluster variety is

$$\mathcal{A}(\mathbb{Z}^T)/\mathcal{A}(\mathbb{Q}^T)/\mathcal{A}(\mathbb{R}^T) \quad \text{resp.} \quad \mathcal{X}(\mathbb{Z}^T)/\mathcal{X}(\mathbb{Q}^T)/\mathcal{X}(\mathbb{R}^T),$$

where $\mathbb{Z}^T = (\mathbb{Z}, \max, +)/\mathbb{Q}^T = (\mathbb{Q}, \max, +)/\mathbb{R}^T = (\mathbb{R}, \max, +)$.

Cluster duality and the Fock–Goncharov conjecture

Recall: T_N has dual torus T_M and $T_M(\mathbb{Z}^T) = M \otimes_{\mathbb{Z}} \mathbb{Z} = M$ parametrizes a basis of regular functions $\Gamma(T_N, \mathcal{O}_{T_N})$ $\rightsquigarrow$ *characters of T_N*

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Fock–Goncharov conjecture

The tropical cluster variety $\mathcal{X}(\mathbb{Z}^T)$, respectively $\mathcal{A}(\mathbb{Z}^T)$, parametrizes a basis for $\Gamma(\mathcal{A}, \mathcal{O}_{\mathcal{A}})$, respectively $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$.

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false in general (counter examples due to Gross–Hacking–Keel),
true in interesting examples like cluster varieties inside the Grassmannians, flag varieties, configuration space.

Assumption: the full Fock–Goncharov conjecture holds for $\mathcal{A}$, that is $\Theta := \{\vartheta_m : m \in \mathcal{X}(\mathbb{Z}^T)\}$ is a basis for $\Gamma(\mathcal{A}, \mathcal{O}_{\mathcal{A}})$, called *theta basis*.

Wall and chamber structure on $\mathcal{X}(\mathbb{R}^T)$

Fact: for every seed $s' = (e'_1, \dots, e'_n)$ with dual basis $f'_1, \dots, f'_n \in M$ we have $z^{m_1 f'_1 + \cdots + m_n f'_n} \in \Theta$ with $m_i \in \mathbb{N}$ called *cluster monomials* and

$$\mathrm{g}_{s'}(z^{m_1 f'_1 + \cdots + m_n f'_n}) = m_1 f'_1 + \cdots + m_n f'_n$$

[1] conjectured by [FZ], partial results due to [CIKLP], full generality [GHKK]

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Let $\mathcal{G}_{s_0}(s') = \mu^*_{s_0,s'}(\langle f_1', \dots, f_n' \rangle_{\geq 0})$ then $\bigcup_{s' \sim s_0} \mathcal{G}_{s_0}(s')$ is a *simplicial fan*[1]

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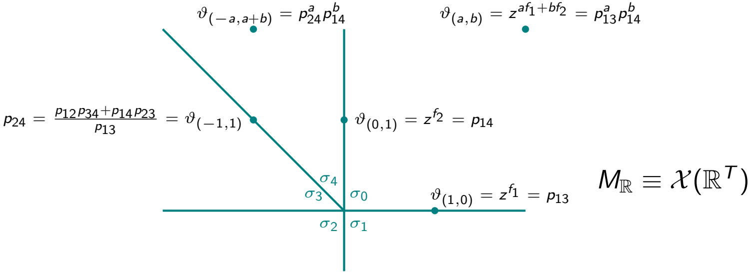
Example: In case of $\mathcal{A} \subset \mathrm{Gr}_2(\mathbb{C}^5)$, so $N = \mathbb{Z}^{\binom{5}{2}}$, consider a slice of M of points $af_{13} + bf_{14}$, $a, b \in \mathbb{Z}$:



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Part II: Compactifying $\mathcal{A}$-cluster varieties

In the initial data $s_0 = \{e_1, \ldots, e_n\} \subset N$ declare $e_k, \ldots, e_n$ *frozen*, i.e. never mutate at $e_k, \ldots, e_n$, then allow vanishing of $z^{f_k}, \ldots, z^{f_n}$.

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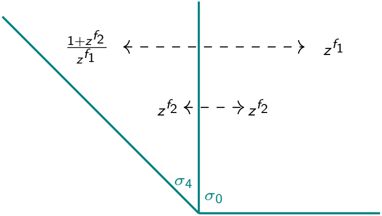

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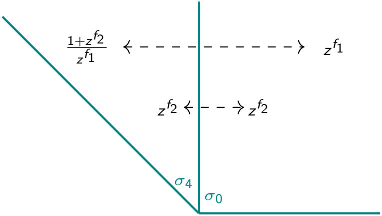

The (partial) compactification $\overline{\mathcal{A}}$ is glued from two copies of $\mathbb{C}^* \times \mathbb{C}$ along the biggest open subset where mutation is still defined.

$D := \overline{\mathcal{A}} \setminus \mathcal{A}$ is called the *boundary divisor*.

Potentials for cluster varieties

In the spirit of mirror symmetry the compactification $\overline{\mathcal{A}}$ of $\mathcal{A}$ should induce a *potential function* on the dual $\mathcal{X}$ cluster variety

Recall: $D = D_k \cup \cdots \cup D_n$ with $D_i = \{z^{f_i} = 0\}$.

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Mild assumptions $\Rightarrow$ may identify tropical points with divisorial discrete valuations:

$$\mathrm{ord}_{D_i} \longleftrightarrow n_i \in \mathcal{A}(\mathbb{Z}^T) \longleftrightarrow \vartheta_i : \mathcal{X} \to \mathbb{C}$$

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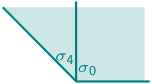
Note: 

$= \{m \in M_{\mathbb{R}} : \langle m, -e_1\rangle \leq 0, \langle m, -e_1 - e_2\rangle \leq 0\}$.

Valuations for cluster varieties

Recall: $\mathcal{X}(\mathbb{Z}^T)|_{T_{M,s}} \equiv M$ and the theta basis $\Theta = \{\vartheta_m : m \in \mathcal{X}(\mathbb{Z}^T)\}$.

Be $(\overline{\mathcal{A}}, D)$ a (partially) compactified cluster variety with theta potential $W : \mathcal{X} \to \mathbb{C}$ and its *tropicalization* $\Xi := \{m \in \mathcal{X}(\mathbb{Z}^T) : W^T(m) \leq 0\}$.

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Given the combinatorial and mild assumptions, we have

$$\overline{\Theta} := \{\vartheta_m : m \in \Xi\} \quad \text{is a basis for} \quad \Gamma(\overline{\mathcal{A}}, \mathcal{O}_{\overline{\mathcal{A}}}).$$

Let $\Xi_s := \text{Cone}(\Xi \cap \mathcal{X}(\mathbb{Z}^T)|_{T_{M,s}}) \subset M_{\mathbb{R}}$.

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Proposition (B.–Cheung–Magee–Nájera Chávez)

Given the above assumptions the assignment $\vartheta_m \mapsto m \in \Xi_s$ for $m \in \Xi_s \cap M$ extends to a valuation

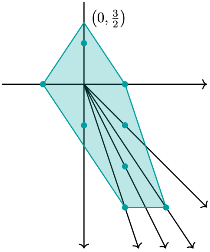
$$\mathrm{g}_s : \Gamma(\overline{\mathcal{A}}, \mathcal{O}_{\overline{\mathcal{A}}}) \setminus \{0\} \to M$$

whose *Newton–Okounkov cone* is Ξ_s.

Part III: Broken line convexity

$\mathcal{X}(\mathbb{Z}^T)|_{T_{M,s}} \equiv M$ is non-canonical as $\mathcal{X}(\mathbb{Z}^T)$ is not a lattice.

But $\mathcal{X}(\mathbb{R}^T)$ has a *wall and chamber structure* and notion of *broken lines*[2].



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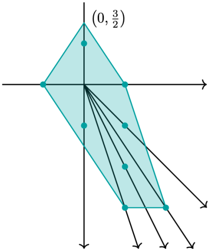

[Cheung–Magee–Nájera Chávez] introduce *broken line convexity*: a closed set $S \subset \mathcal{X}(\mathbb{R}^T)$ is broken line convex iff $\forall a, b \in S$ and any broken line segment ℓ between a, b we have $\ell \subset S$.

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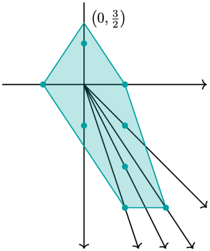

[Cheung–Magee–Nájera Chávez] introduce *broken line convexity*: a closed set $S \subset \mathcal{X}(\mathbb{R}^T)$ is broken line convex iff $\forall a, b \in S$ and any broken line segment ℓ between a, b we have $\ell \subset S$.

Theorem (Cheung–Magee–Nájera Chávez)

A compact set $S \subset \mathcal{X}(\mathbb{R}^T)$ that is broken line convex defines a projective compactification of an $\mathcal{A}$-cluster variety whose graded ring has a theta basis.

[2] combinatorial gadgets replacing pseudoholomorphic disks, their counts give the structure constants for the theta basis

Intrinsic Newton–Okounkov body

Assuming the full Fock–Goncharov conjecture holds, for $f \in \Gamma(\mathcal{A}, \mathcal{O}_\mathcal{A})$ we have $f = \sum_{m \in \mathcal{X}(\mathbb{Z}^T)} c_m \vartheta_m$ and define its *ϑ-Newton polytope*:

$$\mathrm{New}_\vartheta(f) := \mathrm{conv}_{BL}\left(m \in \mathcal{X}(\mathbb{Z}^T) : c_m \neq 0\right) \subset \mathcal{X}(\mathbb{R}^T).$$

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For $\mathcal{L}$ and line bundle on $\overline{\mathcal{A}}$ and $R(\mathcal{L}) = \bigoplus_{j \geq 0} R_j(\mathcal{L})$ its section ring we define the *intrinsic Newton–Okounkov body*

$$\Delta_{BL}(\mathcal{L}) := \overline{\mathrm{conv}_{BL}\left(\bigcup_{j \geq 1} \left(\bigcup_{f \in R_j(\mathcal{L})} \left\{ \frac{1}{j} \mathrm{New}_{\vartheta}(f) \right\} \right) \right)} \subset \mathcal{X}(\mathbb{R}^T)$$

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Theorem (B.–Cheung–Magee–Nájera Chávez)

For a seed s and $\mathrm{g}_s : R(\mathcal{L}) \setminus \{0\} \to M \equiv \mathcal{X}(\mathbb{Z}^T)|_{T_{M,s}}$ we have

$$\Delta_{BL}(\mathcal{L})|_{T_{M,s}} = \Delta(\mathrm{g}_s, \mathcal{L}) = \Xi_s \cap H_{\mathcal{L}} \subset M_{\mathbb{R}}$$

$\rightsquigarrow$ the broken line convex hull *detects missing vertices*.

Example: Grassmannian

For a general Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$ we define the initial seed s with basis $\{e_{i\times j}\}_{i,j}$ where $z^{f_{i\times j}} := p_{[1,k-j]\cup[k-j+i+1,k+i]}$ and $\{\cdot,\cdot\}$ given by:

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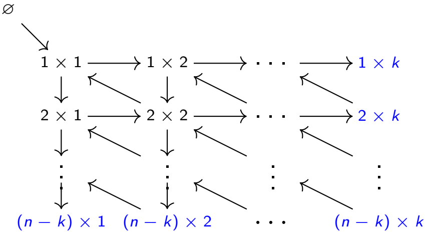

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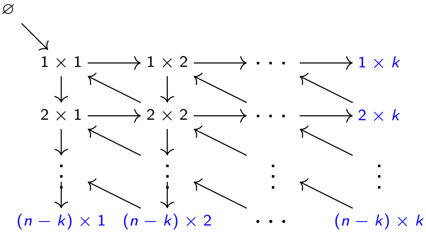
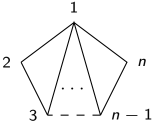


Exercise: for $k = 2$

Q corresponds to



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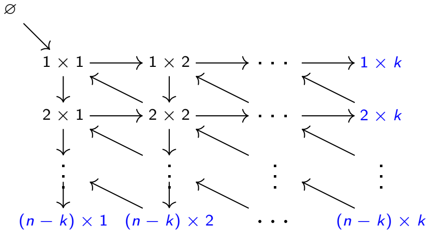

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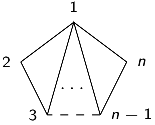

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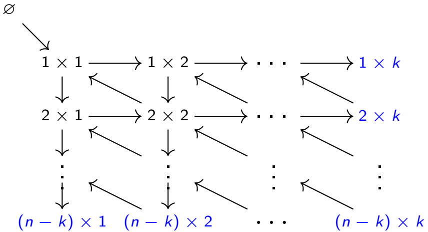

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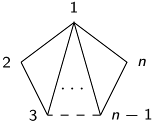

Mutation at 4-valent vertices $\leftrightarrow$ 3-term Plücker relations, e.g.

$$\mu_{1\times 1}(z^{f_{1\times 1}}) = \frac{z^{f_{1\times 2}+f_{2\times 1}} + z^{f_{\varnothing}+f_{2\times 2}}}{z^{f_{1\times 1}}}$$

$$= \frac{p_{[1,k-2]\cup[k,k+1]}\,p_{[1,k-2]\cup[k+1,k+2]} + p_{[1,k-1]\cup\{k+1\}}\,p_{[1,k]}}{p_{[1,k-1]\cup\{k+1\}}} = p_{[1,k-2]\cup\{k,k+2\}}.$$

Part IV: Examples

The *Grassmannian* $\mathrm{Gr}_{k,n}$ is a compactification of a cluster variety [Scott]. More precisely, there exists $(\overline{\mathcal{A}}, D)$ satisfying all above conjectures and assumptions with $\Gamma(\overline{\mathcal{A}}, \mathcal{O}_{\overline{\mathcal{A}}}) \cong \mathrm{Cox}(\mathrm{Gr}_{k,n})$.

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[Marsh–Rietsch] prove a mirror symmetry conjecture for $\mathrm{Gr}_{k,n}$ using their *Plücker potential* $P : \mathrm{Gr}^\circ_{n-k,n} \to \mathbb{C}$.

[Rietsch–Williams] use the cluster structure to define full rank valuations $\mathrm{val}_s : \mathrm{Cox}(\mathrm{Gr}_{k,n}) \setminus \{0\} \to N$ and Newton–Okounkov polytopes $\Delta(\mathrm{val}_s)$ for every seed s. They show

$$\Delta(\mathrm{val}_s) \cong \Gamma_s(P)$$

where $\Gamma_s(P) \subset N \otimes_{\mathbb{Z}} \mathbb{R}$ is the tropicalization of $P|_{T_{N,s}}$.

Example: $\mathrm{Gr}_2(\mathbb{C}^5)$

In the initial chart
$P : \mathrm{Gr}^{\circ}_{3,5} \to \mathbb{C}$ is given by

$$P = \frac{p_{235}}{p_{234}} + \frac{p_{134}}{p_{345}} + \frac{p_{245}}{p_{145}} + \frac{p_{135}}{p_{125}} + \frac{p_{124}}{p_{123}}.$$

In this case,

$$\mathrm{conv}(\{\mathrm{val}_s(p_{ij})\}_{i,j}) = \Delta(\mathrm{val}_s) = \Gamma_s(P)$$

is the *Gelfand–Zetlin polytope* for SL_5 and weight ω_2.

Table 2. The valuations $\mathrm{val}_G(P_J)$ of the Plücker coordinates.

Plücker						
$P_{1,2}$	0	0	0	0	0	0
$P_{1,3}$	1	0	0	0	0	0
$P_{1,4}$	1	1	0	0	0	0
$P_{1,5}$	1	1	1	0	0	0
$P_{2,3}$	1	0	0	1	0	0
$P_{2,4}$	1	1	0	1	0	0
$P_{2,5}$	1	1	1	1	0	0
$P_{3,4}$	2	1	0	1	1	0
$P_{3,5}$	2	1	1	1	1	0
$P_{4,5}$	2	2	1	1	1	1

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Plücker	$(3,3)$	$(2,2)$	$(1,1)$	(3)	(2)	(1)
$P_{1,2}$	0	0	0	0	0	0
$P_{1,3}$	1	0	0	0	0	0
$P_{1,4}$	1	1	0	0	0	0
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Recall, $N = \mathbb{Z}^7$ with basis $\{e_{12}, e_{13}, e_{14}, e_{15}, e_{23}, e_{34}, e_{45}\}$ and $z^{f_{ij}} = p_{ij}$.
For the same initial seed we have $W : \widetilde{\mathcal{X}} \to \mathbb{C}$ given by

$$z^{-e_{15}} + z^{-e_{23}} + z^{-e_{12}}(1 + z^{-e_{13}}) + z^{-e_{45}}(1 + z^{-e_{14}}) + z^{-e_{34}}(1 + z^{-e_{13}}(1 + z^{-e_{14}})).$$

And $\Xi_s \cap (f_{12} + f_{13} + f_{14} + f_{15} + f_{23} + f_{34} + f_{45})^{\perp} \cong \Gamma_s(P)$.

Grassmannians

Remark: The equality $\mathrm{conv}(\{\mathrm{val}_s(p_{ij})\}_{i,j}) = \Delta(\mathrm{val}_s)$ is *false* in general (counterexamples for $\mathrm{Gr}_{3,6}$ and larger). In general it is a hard (nonetheless desirable) to find vertices of a Newton–Okounkov polytope, but broken line convexity helps to do exactly this.

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Fact: For all $\mathcal{A}$- and $\mathcal{X}$-cluster varieties there exists a family of *cluster ensemble maps* $p : \mathcal{A} \to \mathcal{X}$ given by $p_s^* : N \to M$ for every seed s.

$$\begin{array}{ccccccc} \widetilde{\mathrm{Gr}_{k,n}} & \supset & \widetilde{\mathcal{A}} & \xrightarrow{\widetilde{p}} & \widetilde{\mathcal{X}} & \subset & \widetilde{\mathrm{Gr}_{k,n}} \\ \downarrow & & & & & & \downarrow \\ \mathrm{Gr}_{k,n} & \supset & \mathcal{A} & \xrightarrow{p} & \mathcal{X} & \subset & \mathrm{Gr}_{k,n} \end{array} \qquad \begin{array}{ccccccc} \widetilde{\mathrm{Gr}_{n-k,n}} & \supset & \widetilde{\mathcal{A}}^\vee & \xrightarrow{\widetilde{p}^\vee} & \widetilde{\mathcal{X}}^\vee & \subset & \widetilde{\mathrm{Gr}_{n-k,n}} \\ \downarrow & & & & & & \downarrow \\ \mathrm{Gr}_{n-k,n} & \supset & \mathcal{A}^\vee & \xrightarrow{p^\vee} & \mathcal{X}^\vee & \subset & \mathrm{Gr}_{n-k,n} \end{array}$$

We have theta potentials $W : \widetilde{\mathcal{X}} \to \mathbb{C}$ and $W^\vee : \widetilde{\mathcal{X}}^\vee \to \mathbb{C}$ and Plücker potentials $P : \mathcal{A}^\vee \to \mathbb{C}$ and $P^\vee : \mathcal{A} \to \mathbb{C}$.

Grassmannians

Theorem (B.–Cheung–Magee–Nájera Chávez)

For $\widetilde{\mathcal{A}} \subset \widetilde{\mathrm{Gr}_{k,n}}$ there exists a unique *cluster ensemble map* $p : \widetilde{\mathcal{A}} \to \widetilde{\mathcal{X}}$ that pulls back the theta to the Plücker potential $p^*(W) = \widetilde{P}^\vee$.

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Moreover, the dual map $(p^*)_s^\vee : N \to M$ satisfies for Plücker coordinates p_I

$$(p^*)_s^\vee(\nu_s(p_I)) = \mathrm{g}_s(p_I) + c,$$

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and for all seeds s the Newton–Okounkov body $\Delta(g_s)$ is

$$\Delta(g_s) = \mathrm{conv}_{BL}\left(\mathrm{g}_s(p_I) : I \in \binom{[n]}{k}\right)\Big|_{T_M,s} \subset M_{\mathbb{R}}.$$

Corollary: $(p^*)^\vee_s(\Gamma_s) = \Xi_s \cap H = \Delta(\mathrm{g}_s) = (p^*)^\vee_s(\Delta(\nu_s))$, so $\Gamma_s = \Delta(\nu_s)$.

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