Cluster algebras for scattering amplitudes

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Motivation: scattering amplitudes

In quantum field theory physicist compute probabilities of particle interactions via scattering amplitudes.

The *scattering amplitud* is a function on the kinematic space that models the particle configuration, typically a generalized polylogarithm expressed as an iterated integral.

I want to tell you:

1. What kind of functions?
2. Defined on which spaces?
3. Where are the cluster algebras?

Goncharov polylogarithms [C. Duhr, H. Gangl, J.Rhodes JHEP10(2012)075]

For $n \geq 0$ and $a_i \in \mathbb{C}$ we define the *Goncharov polylogarithm* as

$$G(a_1, \ldots, a_n; z) := \int_0^z \frac{dt}{t - a_1} G(a_2, \ldots, a_n; t), \quad G(; z) = 1 \ (G(0) = 0)$$

with $G(k\vec{a}; kx) = G(\vec{a}; x)$ for $a_n \neq 0, k \in \mathbb{C}^*$

Examples:

$$G(\vec{0}_n; x) = \frac{1}{n!} \log^n x, \qquad G(\vec{a}_n; x) = \frac{1}{n!} \log^n \left(1 - \frac{x}{a}\right),$$
$$G(\vec{0}_{n-1}, a; x) = -\mathrm{Li}_n\left(\frac{x}{a}\right), \qquad G(\vec{0}_n, \vec{a}_p; x) = (-1)^p S_{n,p}\left(\frac{x}{a}\right),$$

where $S_{n,p}$ denotes the Nielson polylogarithm.

Symbol calculus [C. Duhr, H. Gangl, J.Rhodes JHEP10(2012)075]

The differential structure

$$dG(a_{n-1},\ldots,a_1;a_n)=\sum_{i=1}^{n-1}G(a_{n-1},\ldots,\hat{a}_i,\ldots,a_1;z)d\log\frac{a_i-a_{i+1}}{a_i-a_{i-1}}$$

motivates the definition of the *symbol map*

$$\mathcal{S}(G(a_{n-1},\ldots,a_1;a_n))=\sum_{i=1}^{n-1}\mathcal{S}(G(a_{n-1},\ldots,\hat{a}_i,\ldots,a_1;z))\otimes\frac{a_i-a_{i+1}}{a_i-a_{i-1}}$$

applied iteratively it associates a tensor product (called *word*) in variables (called *letters*).

Distributivity:

$$C\otimes(a\cdot b)\otimes D=C\otimes a\otimes D+C\otimes b\otimes D$$

$$C\otimes a^n\otimes D=n\big(C\otimes a\otimes D\big)\,,\qquad n\in\mathbb{Z},$$

First examples [C. Duhr, H. Gangl, J.Rhodes JHEP10(2012)075]

Example 0:

$$\mathrm{Li}_k(z) = \int_0^z \mathrm{Li}_{k-1}(t) d\log t, \quad \mathrm{Li}_1(z) = -\log(1-z) \tag{15}$$

clearly have

$$\mathrm{symbol}(\mathrm{Li}_k(z)) = -(1-z) \otimes \underbrace{z \otimes \cdots \otimes z}_{k-1 \text{ times}}. \tag{16}$$

Example 1: For a, b distinct and non zero one verifies

$$G(a,b;x) = \mathrm{Li}_2\left(\frac{b-x}{b-a}\right) - \mathrm{Li}_2\left(\frac{b}{b-a}\right) + \log\left(1-\frac{x}{b}\right)\log\left(\frac{x-a}{b-a}\right)$$

Combined with **Example 0** we may deduce:

$$\mathcal{S}(G(a,b;x)) = \left(1-\frac{x}{b}\right) \otimes \left(1-\frac{x}{a}\right) - \left(1-\frac{x}{b}\right) \otimes \left(1-\frac{b}{a}\right) + \left(1-\frac{x}{a}\right) \otimes \left(1-\frac{a}{b}\right)$$

Example 2: Scattering amplitude

[A. Goncharov, M. Spradlin, C. Vergu, A. Volovich. Phys. Rev. Lett. 105 (2010) 151605]

After manipulating the ($\mathcal{N} = 4$ super Yang-Mills six particle two loop) *scattering amplitude* we are left with the remainder function:

$$R_6^{(2)} = \sum_{i=1}^{3}\left(L_i - \frac{1}{2}\operatorname{Li}_4(-v_i)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\operatorname{Li}_2(-v_i)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72},$$

in terms of the functions

$$L_i = \frac{1}{384}P_i^4 + \sum_{m=0}^{3}\frac{(-1)^m}{(2m)!!}P_i^m(\ell_{4-m}(x_i^+) + \ell_{4-m}(x_i^-)),$$

$$P_i = 2\operatorname{Li}_1(-v_i) - \sum_{j=1}^{3}\operatorname{Li}_1(-v_j),$$

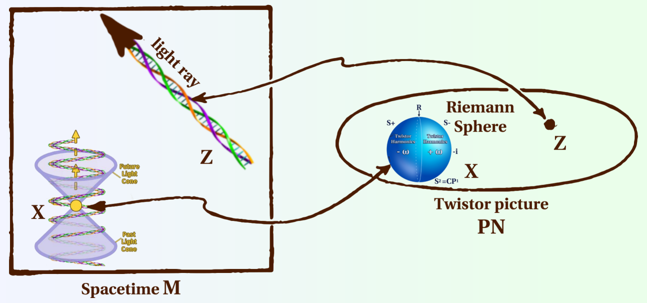
and

$$J = \sum_{i=1}^{3}\ell_1(x_i^+) - \ell_1(x_i^-),$$

$$\ell_n(x) = \frac{1}{2}(\operatorname{Li}_n(-x) - (-1)^n\operatorname{Li}_n(-1/x)).$$

Twistor theory

[R. Penrose. *Twistor algebra*. J. Math. Phys. 8, 345–366 (1967)]



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- conformal invariance and other symmetries become apparent
- differential equations ⇝ complex and algebraic geometry AttiyahWard 1977, PenroseRindler 1984
- scattering amplitudes ⇝ compact expressions Witten 2004

Momentum twistors

[A. Hodges. *Eliminating spurious poles from gauge-theoretic amplitudes* JHEP, 2013(5), 135]

A particle configuration in $\mathcal{N} = 4$ super Yang Mills may be represented by momentum twistors $Z_1, \ldots, Z_n \in \mathbb{C}P^3$. The system is parametrized by

$$\langle ijkl \rangle := \det(Z_i Z_j Z_k Z_l)$$

for $1 \leq i < j < k < l \leq n$ satisfying determinantal identities

$\rightsquigarrow$ closely related to the Grassmannian $\mathrm{Gr}_{4,n}$

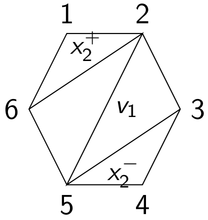
Notation: for $n = 6$ simplify $\langle i, j, k, l \rangle = \langle p, q \rangle$ where $\{i, j, k, l, p, q\} = \{1, \ldots, 6\}$

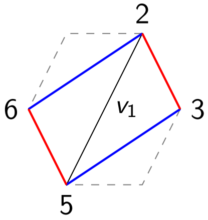
Example 2: symbol letters of $R_6^{(2)}$

[A. Goncharov, M. Spradlin, C. Vergu, A. Volovich. Phys. Rev. Lett. 105 (2010) 151605]

$$
\begin{aligned}
v_1 &= \frac{\langle 35\rangle\langle 26\rangle}{\langle 23\rangle\langle 56\rangle}, & v_2 &= \frac{\langle 13\rangle\langle 46\rangle}{\langle 16\rangle\langle 34\rangle}, & v_3 &= \frac{\langle 15\rangle\langle 24\rangle}{\langle 45\rangle\langle 12\rangle}, \\
x_1^+ &= \frac{\langle 14\rangle\langle 23\rangle}{\langle 12\rangle\langle 34\rangle}, & x_2^+ &= \frac{\langle 25\rangle\langle 16\rangle}{\langle 56\rangle\langle 12\rangle}, & x_3^+ &= \frac{\langle 36\rangle\langle 45\rangle}{\langle 34\rangle\langle 56\rangle}, \\
x_1^- &= \frac{\langle 14\rangle\langle 56\rangle}{\langle 45\rangle\langle 16\rangle}, & x_2^- &= \frac{\langle 25\rangle\langle 34\rangle}{\langle 23\rangle\langle 45\rangle}, & x_3^- &= \frac{\langle 36\rangle\langle 12\rangle}{\langle 16\rangle\langle 23\rangle}
\end{aligned}
$$

Observe that combinatorially the symbol alphabet corresponds to *quadrilaterals in a hexagon:*



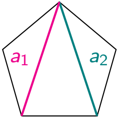


$$v_1 = \frac{\langle 35\rangle\langle 26\rangle}{\langle 23\rangle\langle 56\rangle}$$

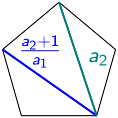
Cluster algebras [S. Fomin, A. Zelevinsky math/0104151]

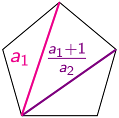
Commutative algebras with special generators called *cluster variables* grouped into *clusters*, and involutions called *mutations* for generating a new cluster from a given one.

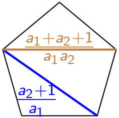
$$a_{i+1} = \frac{a_i+1}{a_{i-1}}$$

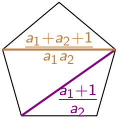


$$\mathcal{A} = \langle a_1, a_2, \frac{a_2+1}{a_1}, \frac{a_1+a_2+1}{a_1 a_2}, \frac{a_1+1}{a_2} \rangle \subset \mathbb{Q}(a_1, a_2)$$









Symbol alphabet and $\mathrm{Gr}_{4,n}$ cluster algebras

In $\mathcal{N} = 4$ super Yang–Mills the symbol alphabet of the scattering amplitude for n particles

$n \leq 7$ $\mathrm{Gr}_{4,n}$ has finitely many cluster variables and they determine the complete symbol alphabet

[Golden, Goncharov, Spradlin, Vergu, Volovich. JHEP 01 (2014) 091]

$n = 8$ $\mathrm{Gr}_{4,8}$ has infinitely many cluster variables, the complete alphabet of 272 letters is determined by a finite collection of cluster variables and expressions obtained as limits along infinite mutation sequences

[Drummond, Foster, Gurdogan, Kalousios. JHEP 11 (2021) 071]

Criticism: $\mathcal{N} = 4$ SYM is a toy model *too far* from the real world modeled by the standard model

Question: Are cluster algebras relevant in other models?

Beyond $\mathcal{N} = 4$ super Yang–Mills

Let $\{p_1, \ldots, p_n\} \subset \mathbb{R}^4$ be a configuration of n lightlike particles in QCD (standard model) twistor theory yields

1. its helicity spinors: n pairs $(\lambda_i, \tilde{\lambda}_i) \in \mathbb{C}^{2\times 2}$ with $\lambda_i \tilde{\lambda}_i^T = 0$
2. define $\Lambda = [\lambda_1, \ldots, \lambda_n], \tilde{\Lambda} = [\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n] \in \mathrm{Gr}_{2;n}$ with $\Lambda \tilde{\Lambda}^T = 0$

Define the *spinor helicity variety*

$$\mathcal{SH}_n := \{(\Lambda, \tilde{\Lambda}) \in \mathrm{Gr}_{2;n} \times \mathrm{Gr}_{2;n} : \Lambda \tilde{\Lambda}^T = 0\}.$$

It inherits a cluster algebra from partial flag varieties

[L.B., J.-R. Li, arXiv:2408.14956[math.AG]]

Symbol alphabet for 5 particles in QCD

[L.B., J. Drummond, R. Glew. JHEP11(2023)002]

Analytic computations have found a symbol alphabet of 31 letters, one of which has unclear physical interpretation

[T. Gehrmann, J. M. Henn and N. A. Lo Presti. Phys. Rev. Lett. 116 (2016) 062001]

[D. Chicherin, J. Henn and V. Mitev. JHEP 05 (2018) 164]

Reformulated in spinor helicity variables up to cyclic symmetry it is

$$W_1 = \langle 12\rangle[12], \qquad W_6 = \langle 34\rangle[34] + \langle 45\rangle[45],$$
$$W_{11} = \langle 34\rangle[34] + \langle 35\rangle[35], \qquad W_{16} = \langle 13\rangle[13],$$
$$W_{21} = \langle 13\rangle[13] + \langle 34\rangle[34], \qquad W_{26} = \frac{\langle 45\rangle[51]\langle 12\rangle[24]}{[45]\langle 51\rangle[12]\langle 24\rangle},$$
$$W_{31} = [45]\langle 51\rangle[12]\langle 24\rangle - \langle 45\rangle[51]\langle 12\rangle[24].$$

$$\begin{aligned} W_6 &= \langle 34\rangle[34] + \langle 45\rangle[45] = \langle 12\rangle[12] - \langle 35\rangle[35], \\ W_{11} &= \langle 34\rangle[34] + \langle 35\rangle[35] = \langle 12\rangle[12] - \langle 45\rangle[45], \\ W_{21} &= \langle 13\rangle[13] + \langle 34\rangle[34] = \langle 14\rangle[14] - \langle 25\rangle[25], \end{aligned}$$

All letters but W_{31} are (products of) cluster variables + permutation copies!

Outlook

- Cluster algebras predict adjacencies of symbol words
  [J.Drummond, J. Foster, Ö. Gürdoğan, Phys.Rev.Lett. 120 (2018)]
  [J. Drummond, J. Foster, Ö. Gürdoğan, C. Kalousios, JHEP 11 (2021)]
- 6 particles in QCD
  [A. Pokraka, M. Spradlin, A. Volovicha, H.-C. Weng, arxiv:2506.11895],
  [L.B., J. Drummond, E. Glew, Ö. Gürdoğan, and R. Wright. arxiv:2507.01015]
- Predicting the symbol to bootstrap the amplitude
  [S. Caron-Huot, L. Dixon , A. McLeod, M. von Hippel, Phys.Rev.Lett. 117 (2016)]
  [S. Caron-Huot, L. Dixon, J. Drummond, F. Dulat, J. Foster, PoS CORFU2019 (2020)]

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