Introduction to Spherical Varieties

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This talked is based on the minicourse "Introduction to spherical varieties over the complex numbers" given by Michel Brion in Klöster Heiligkreuztal on March 17-25, 2014. References can be found at

a) Nicolas Perrin: Introduction to spherical varieties,
`http://relaunch.hcm.uni-bonn.de/fileadmin/perrin/spherical.pdf`

b) Nicolas Perrin: On the geometry of spherical varieties,
`http://reh.math.uni-duesseldorf.de/~perrin/survey.pdf`

c) Michel Brion: Spherical varieties,
`http://www-fourier.ujf-grenoble.fr/~mbrion/notes_bremen.pdf`

1 Introduction and examples

Thoughout this talk, G will be a connected reductive group over the complex numbers, and B a Borel subgroup (i. e. a maximal solvable subgroup); the main example to think about is when G is $GL_n(\mathbb{C})$, and B is the subgroup of upper triangular matrices.

Definition 1.1. *Let $H \subseteq G$ be a closed subgroup of G; an **homogeneous space** is the algebraic variety G/H.*

Elements of G/H will be denoted by $aH = \{ah \mid h \in H\}$ where $a \in G$; note that we have an action of G on G/H, such that

$$g \cdot aH = (ga)H = \{gah \mid h \in H\}.$$

Definition 1.2. *An homogenous space G/H is called spherical, if B acts on it with an open orbit.*

The condition on B seems very random; we will see though that it has many deep consequences; for instance, a spherical space can have only finitely many B-orbits. It also has huge consequences in terms of embeddings.

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Definition 1.3. *An equivariant open embedding of a spherical variety $G/H \hookrightarrow X$ with X normal is called* ***spherical embedding****.*

The main result (probably) of the theory of spherical varieties is the complete classification in terms of *coloured fans* of the spherical embeddings, given a spherical variety G/H, due to Luna and Vust [2], that will be the main content of my next talk. Let's now see some examples.

Example 1.4. Toric varieties are a particular case of spherical embeddings. To see this, take G to the the complex torus $(\mathbb{C}^*)^r$; its Borel subgroup is G itself, so G/H is going to be spherical for every choice of H, so that we may very well take as H the trivial subgroups; for this choice, spherical embeddings are equivariant open embeddings

$$(\mathbb{C}^*)^r \hookrightarrow X$$

that are completely classified by the theory of polyhedral fans. The Luna-Vust classification is indeed a generalization of the theory of toric varieties; we will see how in this more general setting, the main philosophy of toric varieties applies - that is, geometric properties of the toric varieties can be read only from the combinatorial data of its polyhedral fan.

Example 1.5. Let P be a parabolic subgroup of G; if G is $GL_n(\mathbb{C})$, you can think at P as any block-upper-triangular subgroup of matrices; homogeneous spaces G/P are called *flag varieties*; these are all spherical varieties, but their theory of embeddings is not particularly interesting because they are already projective.

Example 1.6. Let σ be an involution of G, and $H = G^\sigma$ the subgroup of fixed elements. Then G/H is a particular kind of spherical variety called *symmetric variety*, and Luna-Vust theory again classifies all its spherical embeddings; one particular of these embeddings is going to be the main content of the following talk, the wonderful compactification that De Concini and Procesi describe in [1].

Example 1.7. Consider the space of all smooth quadric hypersurfaces in $\mathbb{P}^n$; this is an homogeneous space by the action of PGL_{n+1}, and is in fact PGL_{n+1}/PO_{n+1}. This is a spherical variety; in fact, it comes as a particular situation of the previous example, considering the involution σ on PGL_{n+1} such that

$$\sigma(A) =^t A^{-1}.$$

This space embeds as an open subset of the space $\mathbb{P}^{\binom{n+2}{n}-1}$ of polynomial of degree 2 in $n+1$ variables; this is an example of spherical embeddings. Another example is the space of complete quadrics, the wonderful compactification, that will (probably) come up the last talk of the day.

Example 1.8. Consider the space of smooth twisted cubic curves in $\mathbb{P}^3$; this can also be seen as an homogeneous space, PGL_4/PGL_2; this is not a spherical variety; in fact, B has dimension 9, while G/H has dimension 12; in this case we say the homogeneous space has complexity 3, because B has orbits at least of codimension 3.

2 Multiplicity free property

Let us now analyze the various properties of spherical varieties. The first one is the *multiplicity free property*; this follows directly from the hypothesis of the open B-orbit. First, let us remind some notions of representation theory.

Remark 2.1. Let V be an finite dimensional irreducible representation of G; in V we have a vector v called the *highest weight vector*, unique up to a constant, such that it is *semiinvariant* for the action of the Borel subgroup B , i,e,

$$Bv \subset \mathbb{C}v, \quad b \cdot v = \lambda(b)v.$$

This defines then a *character* of B, that is a group homomorphism $\lambda : B \to \mathbb{C}^*$, such that $\lambda(b)$ is the only number such that $b \cdot v = \lambda(b)v$. Moreover, the representation V is uniquely determined by the character λ, and for all *dominant* characters λ we have an irreducible representation $V(\lambda)$ of G; the set of dominant characters will be denoted by Λ^+, that is the intersection of a lattice and a convex cone, so that it is going to have a monoid structure.

Let us consider now a representation V of G, that splits in irreducibles

$$V = \bigoplus_{i=1}^{n} V(\lambda_i).$$

Let us consider now the subspace of B-semiinvariant elements $V^{(B)}$; this is going to be generated by all highest weight vectors of the irreducible components $V(\lambda_i)$; so, understanding $V^{(B)}$ and the characters of B induced by its vectors, we can get the whole decomposition in irreducibles of V.

We are ready now to state and prove the first main property of spherical varieties.

Proposition 2.2. *For any spherical variety G/H, considering its coordinate ring $\mathbb{C}[G/H]$ as a G representation, in its decomposition any irreducible representation $V(\lambda)$ of G appears with multiplicity at most 1.*

Proof. Consider now the ring of B semiinvariants

$$\mathbb{C}[G/H]^{(B)} = \bigoplus_{\Lambda^+} \mathbb{C}[G/H]^{(B)}_{\lambda}$$

and we want to prove that every component $\mathbb{C}[G/H]^{(B)}_\lambda$ is one dimensional (remember, its dimension will tell the multiplicity of $V(\lambda)$ in $\mathbb{C}[G/H]$). Suppose f_1 and f_2 belong to the same component $\mathbb{C}[G/H]^{(B)}_\lambda$; then, the quotient f_1/f_2 is going to be a rational function on G/H that is invariant by B, because

$$b \cdot \frac{f_1}{f_2} = \frac{\lambda(b) f_1}{\lambda(b) f_2} = \frac{f_1}{f_2},$$

so it is constant on B orbits; but we have an open (dense) B orbit, so that f_1/f_2 is indeed a constant and $\mathbb{C}[G/H]^{(B)}_\lambda$. □

In fact, a stronger result holds.

Proposition 2.3. *Let X be a normal affine variety with a G-action, then X is a spherical embedding if and only if $\mathbb{C}[X]$ is multiplicity free, i, e,*

$$\mathbb{C}[X] = \bigoplus_{\Lambda(X)} V(\lambda),$$

where $\Lambda(X)$ is a submonoid of Λ^+.

3 G-orbits and B-orbits

Another very important property of X spherical embeddings is about G-orbits.

Proposition 3.1. *Let X be a spherical embedding, then every G orbit in it is spherical. Furthermore, there are only finitely many G-orbits.*

We will (probably) see a sketch of a proof of this result in my next talk.

Example 3.2. Let us consider again the space of twisted cubics curves in space PGL_4/PGL_2, that is not spherical, and its equivariant compactification given by the Kontsevich space of stable maps $\overline{\mathcal{M}}_{0,0}(\mathbb{P}^3, 3)$. Considering the most degenerate situation, that is when the stable map is a degree 3 map onto a line, the stable map is uniquely determined by the image line and the 4 ramification points; on such a data PGL_4 acts with infinitely many orbits.

We can dig even deeper, analyzing B-orbits.

Proposition 3.3. *Every spherical variety G/H contains finitely many B-orbits.*

Example 3.4. These orbits are very important; for instance, in the case of the flag varieties G/P, the B orbits are called *Schubert cycles.*

Corollary 3.5. *Let X be a spherical embedding, then it has finitely many B-orbits.*

References

[1] C. De Concini, C. Procesi, *Complete symmetric varieties*, Invariant Theory, Lecture Notes in Math., 996, Springer-Verlag (1983), pp. 1–44

[2] D. Luna, T. Vust, *Plongements d'espaces homogènes*, Commentarii mathematici Helvetici 58 (1983), 186-245.